FIFTH EDITION

TRIGONOMETRY

Mark Dugopolski

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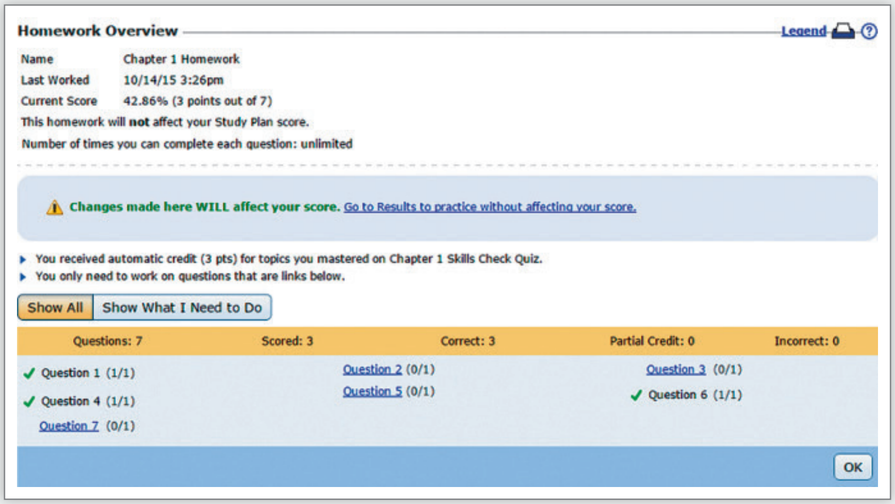

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Trigonometry

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5e Trigonometry

Mark Dugopolski
Southeastern Louisiana University

Pearson

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Courseware Portfolio Management Assistant: Mary Catherine Connors
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PREFACE

The fifth edition of *Trigonometry* is designed for a variety of students with different mathematical needs. For those students who will take additional mathematics, this text will provide the proper foundation of skills, understanding, and insights for success in subsequent courses. For those students who will not further pursue mathematics, the extensive emphasis on applications and modeling will demonstrate the usefulness and applicability of trigonometry in the modern world. I am always on the lookout for real-life applications of mathematics, and I have included many problems that people actually encounter on the job. With an emphasis on problem solving, this text provides students an excellent opportunity to sharpen their reasoning and thinking skills. With increased problem-solving capabilities, students will have confidence to tackle problems that they encounter inside and outside the mathematics classroom.

NEW TO THIS EDITION

- Numerous explanations, examples, and exercises have been rewritten in response to comments from users of the fourth edition.
- This edition contains many new nonstandard exercises, ranging from easy to challenging. There is nothing this author enjoys more than creating original exercises. (e.g. exercise 33, p. 274).
- Section 2.5, Combining Functions, has been rewritten to put more emphasis on producing the graphs with a graphing calculator than drawing them by hand.
- Section 3.2, Verifying Identities, has been rewritten so that there are six examples corresponding to the six-point strategy for verifying identities. The exercise set now contains four exercises corresponding to each point of the six-point strategy and they are labeled as such (e.g. exercise 15, p. 181). This arrangement will give students the opportunity to see how each point of the strategy can be used. In the mixed exercises that follow, students decide which point of the strategy will work on each exercise.
- Multiplying trigonometric binomials and factoring trigonometric expressions have been moved from Section 3.2 to Section 3.1. Now Section 3.2 concentrates solely on verifying identities using the six-point strategy.
- All graphing calculator screen shots have been redone using the TI-84 Plus CE.
- A new section has been added to Chapter 6, Fun with Polar and Parametric Equations. With polar and parametric equations, producing interesting graphs on a calculator that would be impossible to draw by hand. This section contains only one exercise on page 360. This author will award a $100 prize to the first student who submits a correct solution. See page 360 for more details.

CONTINUING FEATURES

With each new edition, all of the features are reviewed to make sure they are providing a positive impact on student success. The continuing features of the text are listed here.

Strategies for Success

- **Chapter Opener** Each chapter begins with chapter opener text that discusses a real-world situation in which the mathematics of the chapter is used. Examples and exercises that relate back to the chapter opener are included in the chapter.
- **Try This** Occurring immediately after every example in the text is an exercise that is very similar to the example. These problems give students the opportunity to immediately check their understanding of the example. Solutions to all *Try*

This exercises are in Online Appendix A, which can be found in MyLab Math or in the Instructor Resource Center. See p. 116.

- **Summaries** of important concepts are included to help students clarify ideas that have multiple parts. See p. 235.
- **Strategies** contain general guidelines for solving certain types of problems. They are designed to help students sharpen their problem-solving skills. See p. 180.
- **Procedures** are similar to *Strategies* but are more specific and more algorithmic. *Procedures* are designed to give students a step-by-step approach for solving a specific type of problem. See p. 127.
- **Function Galleries** Located throughout the text, these function summaries are all gathered at the end of the text. These graphical summaries are designed to help students link the visual aspects of various families of functions to the properties of the functions. See p. 131.
- **Hints** Selected applications include hints that are designed to encourage students to start thinking about the problem at hand. A *Hint* logo **HINT** is used where a hint is given. See p. 218.
- **Graphing Calculator Discussions** Optional graphing calculator discussions have been included in the text. They are clearly marked by graphing calculator icons so that they can be easily skipped if desired. Although the graphing calculator discussions are optional, all students will benefit from reading them. In this text, the graphing calculator is used as a tool to support and enhance conclusions, not to make conclusions. See p. 171.

Section Exercises and Review

- **For Thought** Each exercise set is preceded by a set of ten true/false questions that review the basic concepts in the section, help check student understanding, and offer opportunities for writing and discussion. The answers to all *For Thought* exercises are included in the back of the *Student Edition*.
- **Exercise Sets** The exercise sets range from easy to challenging and are arranged in order of increasing difficulty. Those exercises that require a graphing calculator are optional and are marked with an icon.
- **Writing/Discussion Exercises** These exercises deepen students' understanding by giving them the opportunity to express mathematical ideas both in writing and to their classmates during small group or team discussions. See p. 243.
- **Outside the Box** Found throughout the text, these problems are designed to get students and instructors to do some mathematics just for fun. I enjoyed solving these problems and hope that you will also. The problems can be used for individual or group work. They may or may not have anything to do with the sections in which they are located and might not even require any techniques discussed in the text. So be creative and try thinking *Outside the Box.* See p. 244. The answers are given in the *Annotated Instructor's Edition* only, and complete solutions can be found in the *Instructor's Solutions Manual.* Online Appendix B contains 34 extra *Outside the Box* problems.
- **Pop Quizzes** Included at the end of every section of the text, the *Pop Quizzes* give instructors and students convenient quizzes that can be used in the classroom to check understanding of the basics. The answers appear in the *Annotated Instructor's Edition* only. New for this edition, all Pop Quiz questions are available in Learning Catalytics to assess students in real time! See p. 252.
- **Linking Concepts** This feature is located at the end of nearly every exercise set. It is a multipart exercise or exploration that can be used for individual or group work. The idea of this feature is to use concepts from the current section along with concepts from preceding sections or chapters to solve problems that illustrate the links among various ideas. Some parts of these questions are open-ended and require somewhat more thought than standard skill-building exercises. See p. 215. Answers are given in the *Annotated Instructor's Edition* only, and full solutions can be found in the *Instructor's Solutions Manual.*

Chapter Review

- **Highlights** This end-of-chapter feature contains an overview of all of the concepts presented in the chapter, along with brief examples to illustrate the concepts.
- **Chapter Review Exercises** These exercises are designed to give students a comprehensive review of the chapter without reference to individual sections and to prepare students for a chapter test.
- **Chapter Test** The problems in the *Chapter Test* are designed to measure the student's readiness for a typical one-hour classroom test. Instructors may also use them as a model for their own end-of-chapter tests. Students should be aware that their in-class test could vary from the *Chapter Test* due to different emphasis placed on the topics by individual instructors.
- **Tying It All Together** Found at the end of most chapters in the text, these exercises help students review selected concepts from the present and prior chapters and require students to integrate multiple concepts and skills. See p. 219.
- **Index of Applications** The many applications contained within the text are listed in the *Index of Applications* that appears at the end of the text. The applications are page referenced and grouped by subject matter.

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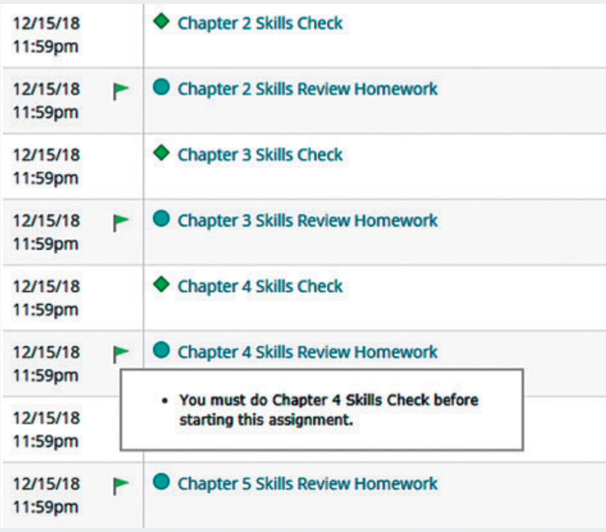

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Resources for Success

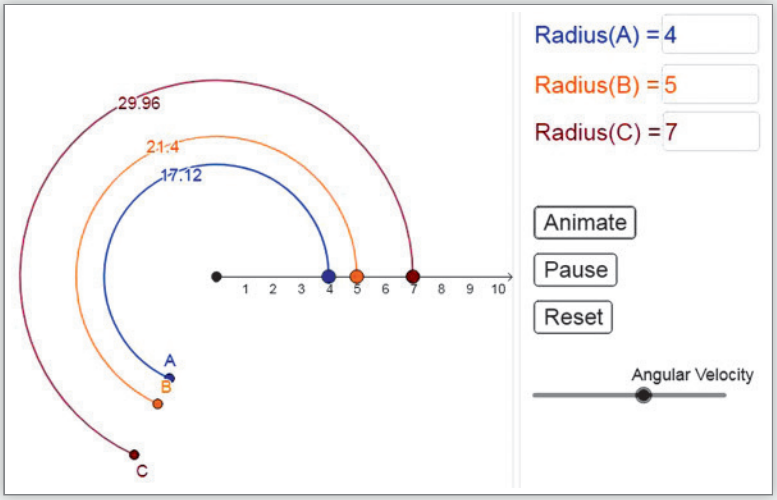
MyLab Math with Integrated Review for *Trigonometry* 5e

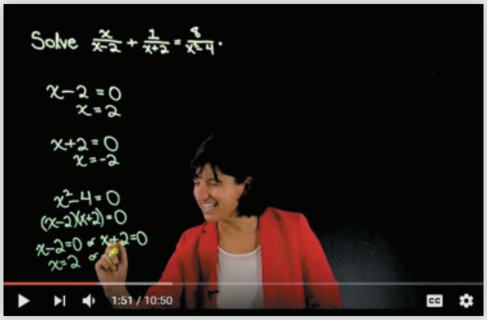
by Mark Dugopolski (access code required)

MyLab Math is tightly integrated with each author's style, offering a range of book-specific multimedia resources so your students have a consistent experience across text and technology.

New! Guided Visualizations

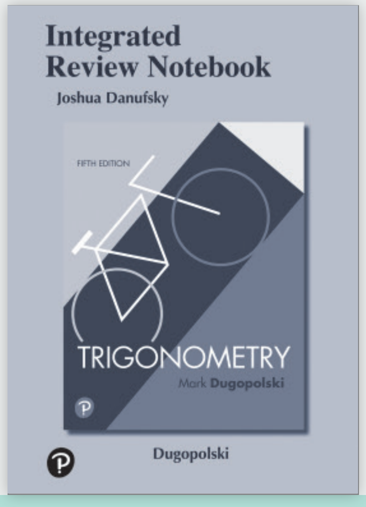
Guided Visualizations bring mathematical concepts to life, helping students visualize the concepts through directed explorations and purposeful manipulation. Guided Visualizations accompany the Integrated Review topics and key concepts from the main text, allowing students to dynamically explore and think more deeply about key ideas in Trigonometry. Guided Visualizations and accompanying assessment exercises can be assigned in MyLab Math.





New! Worked Example Videos

Worked example videos engage and support students outside the classroom, covering all Integrated Review prerequisite topics and key examples from the book, hand-picked by the author. Lightboard technology creates a personal experience and simulates an in-class environment. Accompanying assessment questions in MyLab Math make these brand new videos assignable.



New! Integrated Review Notebook

This printed guide aligns to the Integrated Review topics and gives students a structured place to take notes as they work through the prerequisite objectives. Definitions, unique examples, and important concepts are highlighted. The notebook is available as a print supplement or in MyLab Math for download.

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Powerpoint® Lecture Slides

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Classroom presentation slides are geared specifically to sequence the text. Available in MyLab Math.

Annotated Instructor's Edition

(ISBN: 9780135207468)
This edition provides answers beside the text exercises for most exercises and in an answer section at the back of the book for all others. Groups of exercises are keyed back to corresponding examples from the section.

Instructor's Solutions Manual

(Download Only)
This manual provides complete solutions to all text exercises, including the *For Thought* and *Linking Concepts* exercises.

Instructor's Testing Manual

(Download only)
This resource provides six prepared test forms for each chapter of the text as well as answers. One test form is available for each section of Chapter P.

Student Resources

Additional resources are available to help students succeed.

Example Videos

These new videos provide comprehensive coverage of key topics in the text in an engaging format. Includes optional subtitles in English and Spanish. All videos are assignable within MyLab Math.

Student's Solutions Manual

(ISBN: 9780135232927)
This manual provides detailed solutions to odd-numbered exercises in the text.

Integrated Review Notebook

(ISBN: 9780135207529)
This printed guide offers a structured place to engage with the foundational concepts of Trigonometry. Each concept has a concise exposition followed by a guided example and practice exercises. Ideal for corequisite class time or independent study.

Graphing Resources

Interactive tutorials and how-to videos are available for GeoGebra and TI-84 Plus, respectively. These resources and more can be found in the **Graphing Resources** tab in MyLab Math. Students will be able to launch GeoGebra Graphing Calculator from that tab or they can download the free app while completing their homework assignments.

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Mark Dugopolski
Ponchatoula, Louisiana

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P Algebraic Prerequisites

In the 1995 America's Cup race the defending *Young America* kept up a good face, but from the start it was clear that the New Zealand entry, *Black Magic*, was sailing higher and faster in the 11-knot breeze and rolling swells. For two-and-a-half years the crew had done everything they could to prepare for that moment.

Gone are the days when raw sailing ability and stamina won races like the America's Cup. New Zealand's *Black Magic* sported a mathematically created design that reduced its drag and optimized its stability and speed.

WHAT YOU WILL **LEARN**

Preparation is just as important in trigonometry as it is in sailing. In this chapter we review some of the basics of algebra that are necessary for success in trigonometry. Throughout this chapter, you will see that even basic concepts of algebra have applications in business, science, engineering, and even sailing.

P.1 The Cartesian Coordinate System

Table P.1

Toppings x	Cost y
0	$ 5
1	7
2	9
3	11
4	13

In this chapter we study pairs of variables and how they are related. For example, p might represent the price of gasoline and n the number of gallons that you consume in a month; x might be the number of toppings on a medium pizza and y the cost of that pizza; or h might be the height of a two-year-old child and w the weight. To picture relationships between pairs of variables we use a two-dimensional coordinate system.

A Two-Dimensional Coordinate System

Suppose that the cost of a medium pizza is \$5 plus \$2 per topping. Table P.1 shows a chart in which x is the number of toppings and y is the cost. To indicate that $x = 3$ and $y = 11$ go together, we can use the **ordered pair** (3, 11). The ordered pair (2, 9) indicates that a 2-topping pizza costs \$9. The **first coordinate** of the ordered pair represents the number of toppings and the **second coordinate** represents the cost. The assignment of what the coordinates represent is arbitrary, but once made it is kept fixed. For example, in the present context the ordered pair (11, 3) indicates that an 11-topping pizza costs \$3, and it is not the same as (3, 11). This is the reason they are called "ordered pairs."

Note that parentheses are used to indicate ordered pairs and also to indicate intervals of real numbers. For example, (3, 7) could be an ordered pair or the interval of real numbers between 3 and 7. However, the meaning of this notation should always be clear from the discussion.

Ordered pairs are pictured on a plane in the **rectangular coordinate system,** or **Cartesian coordinate system,** named after the French mathematician René Descartes (1596–1650). The Cartesian coordinate system consists of two number lines drawn perpendicular to one another, intersecting at zero on each number line as in Fig. P.1. The point at which they intersect is called the **origin.** The horizontal number line is the ***x*-axis** and the vertical number line is the ***y*-axis.** The two number lines divide the plane into four regions called **quadrants,** numbered as in Fig. P.1. The quadrants do not include any points on the axes.

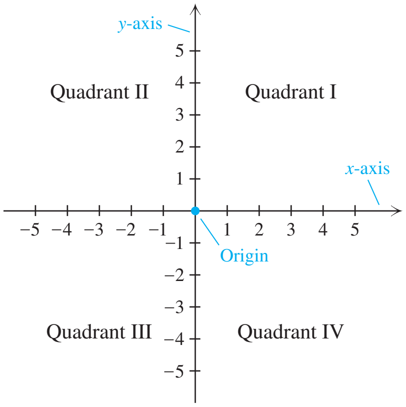


Figure P.1

We call a plane with a rectangular coordinate system the **coordinate plane,** or the ***xy*-plane.** Every ordered pair of real numbers (a, b) corresponds to a point P in the xy-plane. For this reason, ordered pairs of numbers are often called **points.** The numbers a and b are called the **coordinates** of point P. Locating the point P that corresponds to (a, b) in the xy-plane is referred to as **plotting** or **graphing** the point, and P is called the **graph** of (a, b).

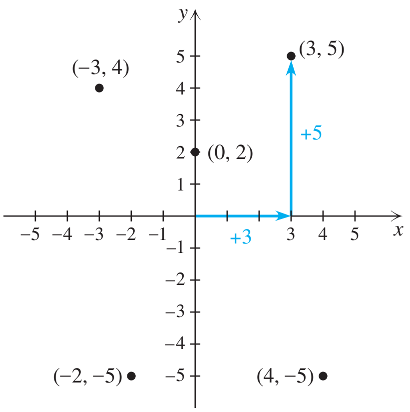


Figure P.2

EXAMPLE 1 Plotting points

Plot the points $(3, 5)$, $(4, -5)$, $(-3, 4)$, $(-2, -5)$, and (0, 2) in the xy-plane.

Solution

The point (3, 5) is located 3 units to the right of the origin and 5 units above the x-axis as shown in Fig. P.2. The point $(4, -5)$ is located 4 units to the right of the origin and 5 units below the x-axis. The point $(-3, 4)$ is located 3 units to the left of the origin and 4 units above the x-axis. The point $(-2, -5)$ is located 2 units to the left of the origin and 5 units below the x-axis. The point (0, 2) is on the y-axis because its first coordinate is zero.

TRY THIS. Plot $(-3, -2)$, $(-1, 3)$, $(5/2, 0)$, and $(2, -3)$.

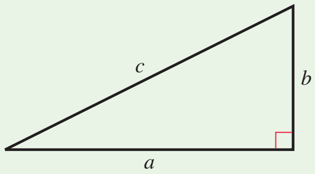
Note that for points in quadrant I, both coordinates are positive. In quadrant II the first coordinate is negative and the second is positive, whereas in quadrant III, both coordinates are negative. In quadrant IV the first coordinate is positive and the second is negative.

The Pythagorean Theorem

To find a formula for the distance between two points in the Cartesian coordinate system we need the Pythagorean theorem. The Pythagorean theorem gives a relationship between the legs and hypotenuse of a right triangle.

The Pythagorean Theorem

In a right triangle the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.



$$a^2 + b^2 = c^2$$

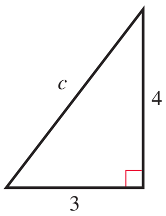
The Pythagorean theorem is probably the most widely known theorem in mathematics. It was known to the Babylonians 4000 years ago and has been proven by such notables as Euclid, Leonardo da Vinci, and James Garfield (the 20th U.S. president). You can find many proofs on the Internet. The first site that this author found contained 54 proofs.

If any two sides of a right triangle are known, the Pythagorean theorem can be used to find the third side, as shown in the next example.

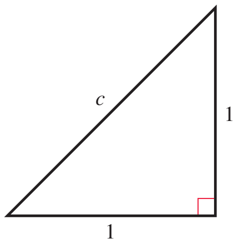
EXAMPLE 2 Applying the Pythagorean theorem

Find the unknown side in each given triangle.

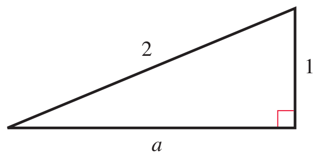
a.



b.



c.



Solution

a. The lengths of the legs are 3 and 4 and the hypotenuse is unknown. Use $c^2 = a^2 + b^2$ with $a = 3$ and $b = 4$:

$$c^2 = 3^2 + 4^2$$

$$c^2 = 25$$

$$c = \pm\sqrt{25} = \pm 5$$

Since the length of the hypotenuse is a positive number, $c = 5$.

b. The lengths of the legs are each 1 unit and the hypotenuse is unknown. Use $c^2 = a^2 + b^2$ with $a = 1$ and $b = 1$:

$$c^2 = 1^2 + 1^2$$
$$c^2 = 2$$
$$c = \pm\sqrt{2}$$

Since the length of the hypotenuse is positive, $c = \sqrt{2}$.

c. The hypotenuse is 2, one leg is 1, and the other leg is unknown. Use $a^2 + b^2 = c^2$ with $c = 2$ and $b = 1$:

$$a^2 + 1^2 = 2^2$$
$$a^2 = 3$$
$$a = \pm\sqrt{3}$$

Since the length of a leg of a triangle is positive, $a = \sqrt{3}$.

TRY THIS. The legs of a right triangle are 2 feet and 3 feet. Find the length of the hypotenuse.

The Pythagorean theorem is still used by builders in squaring up foundations. They know that 3-4-5, 6-8-10, 9-12-15, and so on, are all right triangles. For example, a builder will measure 9 feet and 12 feet (the legs) on two boards that are supposed to meet at a right angle. If the diagonal (or hypotenuse) measures 15 feet, the builder can be certain that the boards meet at a right angle. Builders also use construction calculators that can determine the length of the hypotenuse when the lengths of the two legs are entered.

Simplifying Square Roots

The exact lengths of the missing sides of the triangles in Example 2 are $\sqrt{2}$ and $\sqrt{3}$. Values for these roots, such as 1.414 and 1.732, that you get from a calculator are approximate. Since these numbers are irrational, they can't be expressed exactly in decimal form. In trigonometry we will often encounter square roots in their exact form. There are two rules to remember when working with square roots:

Rule: Product and Quotient

For any real numbers a and b

1. $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ Product rule for square roots

2. $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}} \quad (b \neq 0)$ Quotient rule for square roots

provided that all of the roots are real numbers.

When using square roots to express exact results we use the product and quotient rules to write them in *simplified form*. (Note: In $\sqrt{a}$, the **radicand** is a.)

Definition: Simplified Form for Square Roots

An expression involving a square root is in **simplified form** if it has

1. ***no*** perfect squares as factors of the radicand,
2. ***no*** fractions in the radicand, and
3. ***no*** square roots in a denominator.

Removing a square root from a denominator is called **rationalizing the denominator.** The next example shows how the product and quotient rules are used to rationalize denominators and get radical expressions into simplified form.

EXAMPLE 3 **Simplifying square roots**

Write each expression in simplified form.

a. $\sqrt{20}$ **b.** $\sqrt{\frac{5}{9}}$ **c.** $\sqrt{\frac{32}{5}}$

Solution

a.
$$\begin{aligned} \sqrt{20} &= \sqrt{4 \cdot 5} && \text{Identify 4 as a perfect square factor.} \\ &= \sqrt{4} \cdot \sqrt{5} && \text{Product rule for square roots} \\ &= 2\sqrt{5} && \text{Simplify } \sqrt{4}. \end{aligned}$$

b.
$$\begin{aligned} \sqrt{\frac{5}{9}} &= \frac{\sqrt{5}}{\sqrt{9}} && \text{Quotient rule for square roots} \\ &= \frac{\sqrt{5}}{3} && \text{Simplify } \sqrt{9}. \end{aligned}$$

c.
$$\begin{aligned} \sqrt{\frac{32}{5}} &= \frac{\sqrt{32}}{\sqrt{5}} && \text{Quotient rule for square roots} \\ &= \frac{\sqrt{16} \cdot \sqrt{2}}{\sqrt{5}} && \text{Product rule for square roots} \\ &= \frac{4 \cdot \sqrt{2}}{\sqrt{5}} && \text{Simplify.} \\ &= \frac{4 \cdot \sqrt{2} \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} && \text{Rationalize the denominator.} \\ &= \frac{4\sqrt{10}}{5} && \text{Simplified form} \end{aligned}$$

TRY THIS. Simplify $\sqrt{50}$, $\sqrt{\frac{3}{16}}$, and $\sqrt{\frac{12}{7}}$.

The Distance Formula

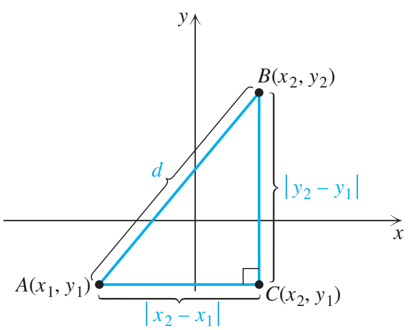


Figure P.3

Consider the points $A(x_1, y_1)$ and $B(x_2, y_2)$ shown in Fig. P.3. Let d represent the length of the line segment $\overline{AB}$. Now $\overline{AB}$ is the hypotenuse of the right triangle in Fig. P.3. The distance between A and C is $|x_2 - x_1|$ and the distance between B and C is $|y_2 - y_1|$. Using the Pythagorean theorem we have

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2.$$

The absolute value symbols can be replaced by parentheses, because $|a - b|^2 = (a - b)^2$ for any real numbers a and b. Since the distance between two points is nonnegative, we have the following formula.

The Distance Formula

The distance d between the points (x_1, y_1) and (x_2, y_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

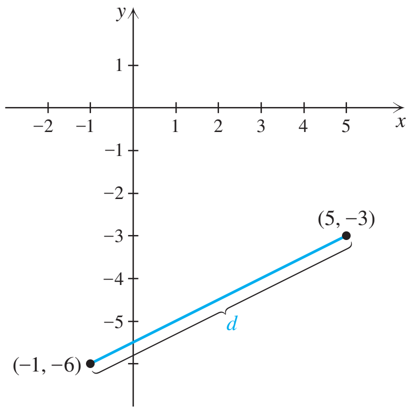


Figure P.4

EXAMPLE 4 Applying the distance formula

Find the distance between $(5, -3)$ and $(-1, -6)$.

Solution

Let $(x_1, y_1) = (5, -3)$ and $(x_2, y_2) = (-1, -6)$. See Fig. P.4. The distance is the same regardless of which point is chosen as (x_1, y_1), or (x_2, y_2). Substitute these values into the distance formula:

$$\begin{aligned} d &= \sqrt{(-1-5)^2 + (-6-(-3))^2} \\ &= \sqrt{(-6)^2 + (-3)^2} \\ &= \sqrt{36+9} = \sqrt{45} = \sqrt{9}\sqrt{5} = 3\sqrt{5} \end{aligned}$$

The exact distance between the points is $3\sqrt{5}$. Note that but $\sqrt{36+9} \neq \sqrt{36} + \sqrt{9}$, but $\sqrt{9 \cdot 5} = \sqrt{9} \cdot \sqrt{5}$.

TRY THIS. Find the distance between $(-3, -2)$ and $(-1, 4)$.

The Midpoint Formula

When you average two test scores (by finding their sum and dividing by 2) you are finding a number midway between the two scores. Likewise, the midpoint of the line segment with endpoints (x_1, y_1) and (x_2, y_2) is found by dividing the sum of the x-coordinates by 2 and the sum of the y-coordinates by 2. See Fig. P.5.

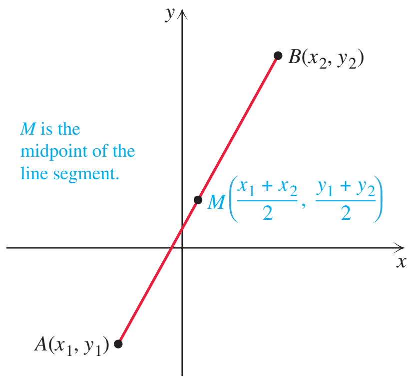


Figure P.5

The Midpoint Formula

The midpoint of the line segment with endpoints (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

The midpoint formula is illustrated in the next example. Note the strange-looking coordinates in part (b). In trigonometry we will often perform arithmetic with the number π.

EXAMPLE 5 Applying the midpoint formula

Find the midpoint of the line segment with the given endpoints.

a. $(1, -3), (5, 4)$ **b.** $\left(\frac{\pi}{2}, 0\right), (\pi, 0)$

Solution

a. To find the midpoint add the coordinates and divide by 2:

$$\left(\frac{1+5}{2}, \frac{-3+4}{2}\right) = \left(3, \frac{1}{2}\right)$$

b. To find the midpoint add the coordinates and divide by 2:

$$\left(\frac{\frac{\pi}{2}+\pi}{2}, \frac{0+0}{2}\right) = \left(\frac{\frac{\pi}{2}+\frac{2\pi}{2}}{2}, 0\right) = \left(\frac{\frac{3\pi}{2}}{2}, 0\right) = \left(\frac{3\pi}{4}, 0\right)$$

TRY THIS. Find the midpoint of the line segment with endpoints $(3, -2)$ and $(5, 6)$.

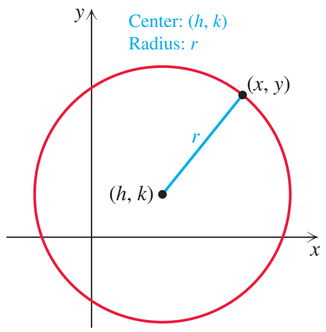


Figure P.6

The Circle

A **circle** is the set of all points in a plane that lie a fixed distance from a given point in the plane. The fixed distance is the **radius** and the given point is the **center.** The distance formula can be used to write an equation for a circle with center (h, k) and radius r $(r > 0)$. A point (x, y) is on the circle shown in Fig. P.6 if and only if it satisfies the equation

$$\sqrt{(x-h)^2 + (y-k)^2} = r.$$

Using the definition of square root, we can write the following equation.

Theorem: Equation for a Circle

The equation for a circle with center (h, k) and radius r for $r > 0$ is

$$(x-h)^2 + (y-k)^2 = r^2.$$

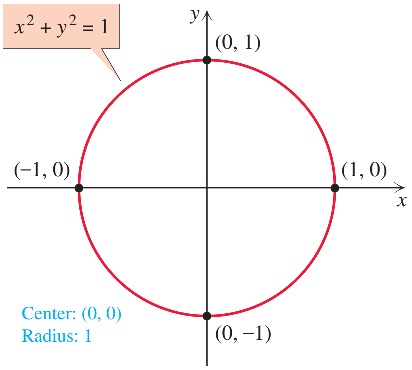


Figure P.7

The form $(x-h)^2 + (y-k)^2 = r^2$ is the **standard form** for the equation of a circle. If (h, k) is $(0, 0)$, then the circle is centered at the origin and its equation is of the form $x^2 + y^2 = r^2$. If $r = 1$, the circle is called a **unit circle.**

EXAMPLE 6 Graphing a circle

Sketch the graph of each circle.

a. $x^2 + y^2 = 1$ **b.** $(x-1)^2 + (y+2)^2 = 9$

Solution

a. The circle has center $(0, 0)$ and radius 1. You can draw the circle as shown in Fig. P.7 with a compass. To draw the circle by hand locate the points that lie 1 unit above, below, right, and left of the center as shown in Fig. P.7. Then sketch a circle through these points.

b. This circle has center $(1, -2)$ and radius 3. Plot the center and the points that lie 3 units above, below, left, and right of the center as shown in Fig. P.8. Then sketch a circle through these points.

To support the conclusions of part (b) with a graphing calculator you must first solve the equation for y:

$$\begin{aligned} (x-1)^2 + (y+2)^2 &= 9 \\ (y+2)^2 &= 9 - (x-1)^2 \\ y + 2 &= \pm\sqrt{9-(x-1)^2} \\ y &= -2 \pm \sqrt{9-(x-1)^2} \end{aligned}$$

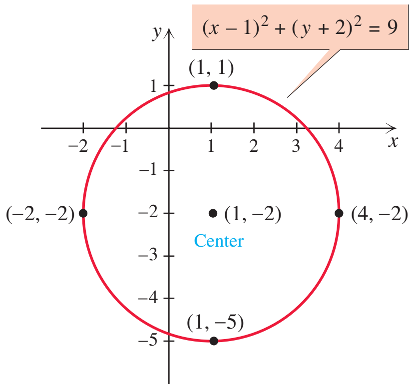
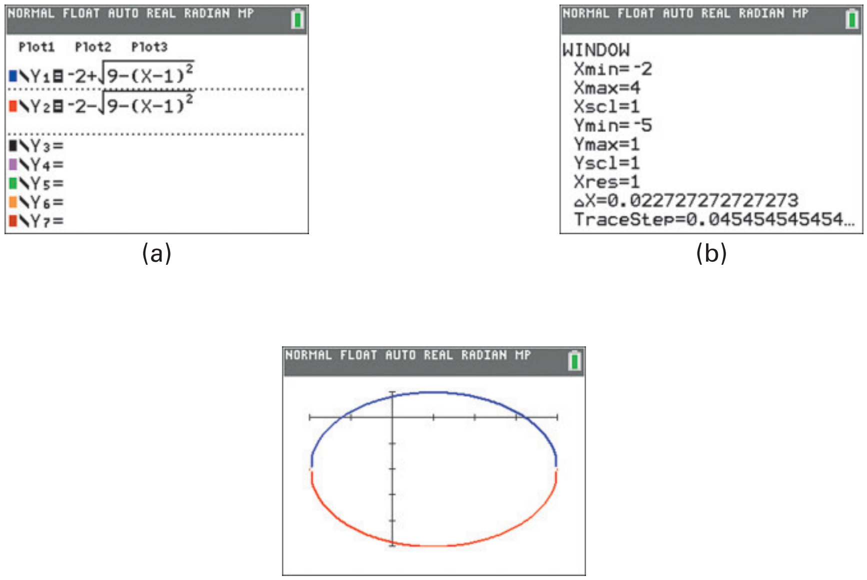


Figure P.8

Now enter y_1 and y_2 as shown in Fig. P.9(a). Set the viewing window as shown in Fig. P.9(b). The graph in Fig. P.9(c) supports our previous conclusions. A circle will look round only if the same unit distance is used on both axes.



(a) (b)

(c)

Figure P.9

TRY THIS. Sketch the graph of $(x + 2)^2 + (y - 4)^2 = 25$.

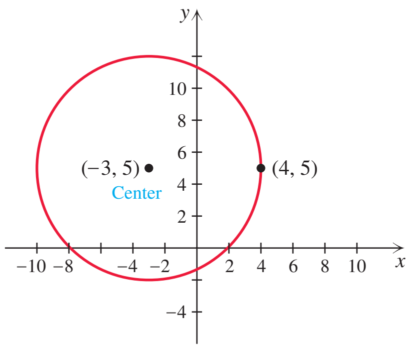


Figure P.10

In the next example we write the equation for a circle from a description of the circle.

EXAMPLE 7 Writing the equation of a circle

Write the standard equation for a circle with center $(-3, 5)$ and passing through $(4, 5)$ as shown in Fig. P.10.

Solution

Since the distance between the center $(-3, 5)$ and $(4, 5)$ is 7 units, the radius of the circle is 7. Use $h = -3$, $k = 5$, and $r = 7$ in the standard equation for a circle:

$$(x - (-3))^2 + (y - 5)^2 = 7^2$$

So the equation of the circle is $(x + 3)^2 + (y - 5)^2 = 49$.

TRY THIS. Write the standard equation for the circle with center $(2, -1)$ and passing through $(3, 6)$.

The Line

Any circle in the coordinate plane is the solution set to an equation of the form $(x - h)^2 + (y - k)^2 = r^2$. Likewise, any straight line in the coordinate plane is the solution set to an equation of another form.

Definition: Linear Equation in Two Variables (Standard Form)

A **linear equation** in two variables x and y is an equation of the form

$$Ax + By = C$$

where A, B, and C are real numbers and A and B are not both zero.

The graph of any linear equation is a straight line. A linear equation can be written in many different forms. The equations

$$x = 5 - y, \qquad y = \frac{1}{2}x - 9, \qquad x = 4, \qquad \text{and} \qquad y = 5$$

are all linear equations because they could all be written in the form $Ax + By = C$.

There is only one line containing any two distinct points in the xy-plane. So to graph a linear equation we need locate only two points that satisfy the equation and draw a line through them. But which points do we use? Since all lines look alike, what distinguishes one line from another is its location. The best way to show the location is to show the points where the line crosses the x- and y-axes. These points are called the ***x*-intercept** and ***y*-intercept.**

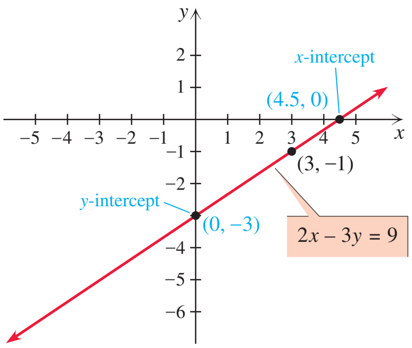


Figure P.11

EXAMPLE 8 Graphing lines and showing the intercepts

Graph each equation. Find and show the intercepts.

a. $2x - 3y = 9$ **b.** $y = 40 - x$

Solution

a. Since the y-coordinate of the x-intercept is 0, we replace y with 0 in the equation:

$$2x - 3(0) = 9$$
$$2x = 9$$
$$x = \frac{9}{2}$$

To find the y-intercept, we replace x with 0 in the equation:

$$2(0) - 3y = 9$$
$$-3y = 9$$
$$y = -3$$

The x-intercept is $(9/2, 0)$ and the y-intercept is $(0, -3)$. Locate the intercepts and draw the line as shown in Fig. P.11. To check, locate a point such as $(3, -1)$, which also satisfies the equation, and see if the line goes through it.

b. If $x = 0$, then $y = 40 - 0 = 40$ and the y-intercept is $(0, 40)$. If $y = 0$, then $0 = 40 - x$ or $x = 40$. The x-intercept is $(40, 0)$. Draw a line through these points as shown in Fig. P.12. Check that $(10, 30)$ and $(20, 20)$ also satisfy $y = 40 - x$ and the line goes through these points.

The calculator graph shown in Fig. P.13 is consistent with the graph in Fig. P.12. Note that the viewing window is set to show the intercepts.

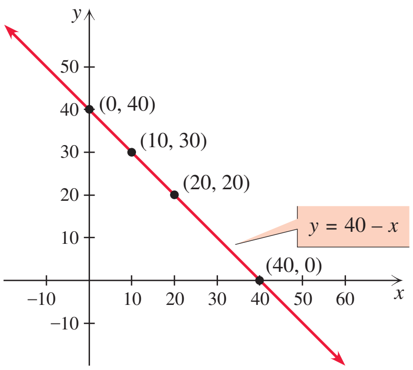


Figure P.12

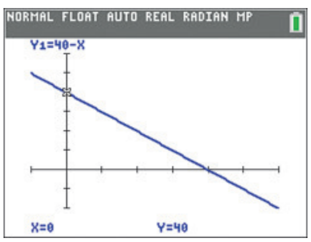


Figure P.13

TRY THIS. Graph $2x + 5y = 10$ and determine the intercepts.

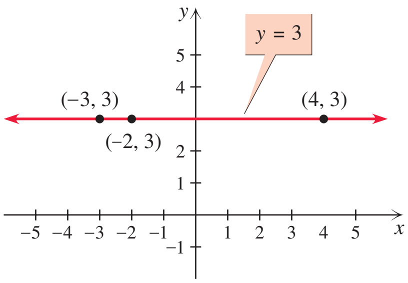


Figure P.14

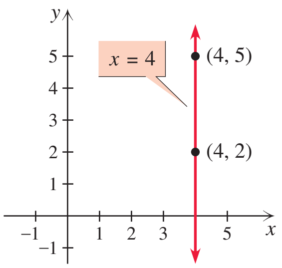


Figure P.15

EXAMPLE 9 Graphing horizontal and vertical lines

Graph each equation in the rectangular coordinate system.

a. $y = 3$ **b.** $x = 4$

Solution

a. The equation $y = 3$ is equivalent to $0 \cdot x + y = 3$. Because x is multiplied by 0, we can choose any value for x as long as we choose 3 for y. So ordered pairs such as $(-3, 3)$, $(-2, 3)$, and $(4, 3)$ satisfy the equation $y = 3$. The graph of $y = 3$ is the horizontal line shown in Fig. P.14.

b. The equation $x = 4$ is equivalent to $x + 0 \cdot y = 4$. Because y is multiplied by 0, we can choose any value for y as long as we choose 4 for x. So ordered pairs such as $(4, -3)$, $(4, 2)$, and $(4, 5)$ satisfy the equation $x = 4$. The graph of $x = 4$ is the vertical line shown in Fig. P.15.

TRY THIS. Graph $y = 5$ in the rectangular coordinate system.

FOR THOUGHT… True or False? Explain.

1. The point $(2, -3)$ is in quadrant II.
2. The point $(4, 0)$ is in quadrant I.
3. The distance between (a, b) and (c, d) is $\sqrt{(a-b)^2 + (c-d)^2}$.
4. The equation $3x^2 + y = 5$ is a linear equation.
5. The graph of $x = 5$ is a vertical line.
6. $\sqrt{7^2 + 9^2} = 7 + 9$.
7. The origin lies midway between $(1, 3)$ and $(-1, -3)$.
8. The distance between $(3, -7)$ and $(3, 3)$ is 10.
9. The x-intercept for the graph of $3x - 2y = 7$ is $(7/3, 0)$
10. The graph of $x^2 + y^2 = 9$ is a circle centered at $(0, 0)$ with radius 9.

P.1 EXERCISES

CONCEPTS

Fill in the blank.

1. If x and y are real numbers, then (x, y) is a(n) __________ pair of real numbers.
2. Ordered pairs are graphed in the rectangular coordinate system or the __________ coordinate system.
3. The horizontal number line in the rectangular coordinate system is the __________.
4. The intersection of the x-axis and the y-axis is the __________.
5. According to the __________, the sum of the squares of the legs in a right triangle is equal to the square of the hypotenuse.
6. The set of all points in a plane that lie a fixed distance from a given point in the plane is a(n) __________.
7. An equation of the form $Ax + By = C$ is a(n) __________ in two variables.
8. The point where a line crosses the y-axis is the __________.

SKILLS

The figure for Exercises 9–18 shows 10 points in the xy-plane. For each point write the corresponding ordered pair and name the quadrant in which it lies or the axis on which it lies.

9. A **10.** B **11.** C **12.** D

13. E **14.** F **15.** G **16.** H

17. I **18.** J

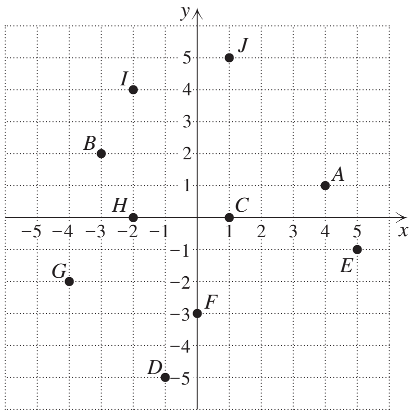
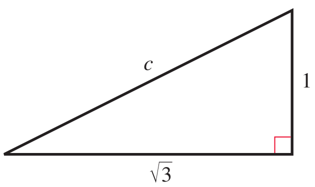


Figure for Exercises 9–18

Find the length of the missing side(s) in each right triangle.

19.



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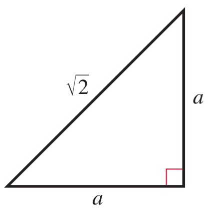

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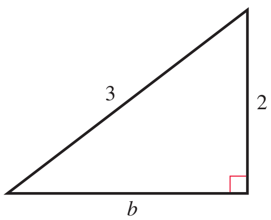

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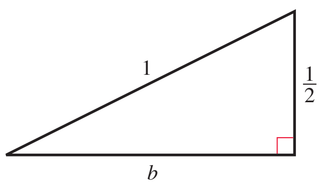

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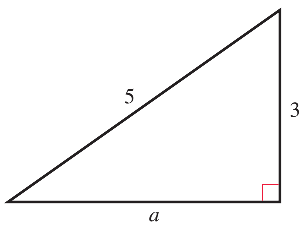

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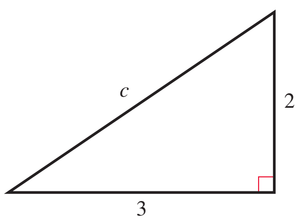


Simplify each expression involving square roots.

25. $\sqrt{28}$ **26.** $\sqrt{50}$ **27.** $\sqrt{\frac{5}{9}}$ **28.** $\sqrt{\frac{3}{16}}$

29. $\sqrt{\frac{2}{3}}$ **30.** $\sqrt{\frac{3}{5}}$ **31.** $\sqrt{\frac{12}{5}}$ **32.** $\sqrt{\frac{25}{3}}$

33. $\frac{1}{\sqrt{3}}$ **34.** $\frac{3}{\sqrt{2}}$ **35.** $\frac{\sqrt{2}}{\sqrt{3}}$ **36.** $\frac{\sqrt{5}}{\sqrt{2}}$

For each pair of points find the distance between them and the midpoint of the line segment joining them.

37. $(1, 3), (4, 7)$ **38.** $(-3, -2), (9, 3)$

39. $(-1, -2), (1, 0)$ **40.** $(-1, 0), (1, 2)$

41. $(0, 0), \left(\sqrt{2}/2, \sqrt{2}/2\right)$ **42.** $(0, 0), \left(\sqrt{3}, 1\right)$

43. $\left(\sqrt{18}, \sqrt{12}\right), \left(\sqrt{8}, \sqrt{27}\right)$

44. $\left(\sqrt{50}, \sqrt{20}\right), \left(\sqrt{72}, \sqrt{45}\right)$

45. $(1.2, 4.8), (-3.8, -2.2)$ **46.** $(-2.3, 1.5), (4.7, -7.5)$

47. $(\pi, 0), (\pi/2, 1)$ **48.** $(0, 0), (\pi/2, 1)$

49. $(\pi, 0), (2\pi, 0)$ **50.** $(\pi, 1), (\pi/2, 1)$

51. $(\pi/3, 1/2), (\pi/2, -1/3)$

52. $(2\pi/3, -1/2), (\pi, -1)$

Determine the center and radius of each circle and sketch the graph.

53. $x^2 + y^2 = 16$ **54.** $x^2 + y^2 = 1$

55. $(x + 6)^2 + y^2 = 36$ **56.** $x^2 = 9 - (y - 3)^2$

57. $(x - 2)^2 = 8 - (y + 2)^2$

58. $(y + 2)^2 = 20 - (x - 4)^2$

Write the standard equation for each circle.

59. Center at $(0, 0)$ with radius $\sqrt{7}$

60. Center at $(0, 0)$ with radius $2\sqrt{3}$

61. Center at $(-2, 5)$ with radius $1/2$

62. Center at $(-1, -6)$ with radius $1/3$

63. Center at $(3, 5)$ and passing through the origin

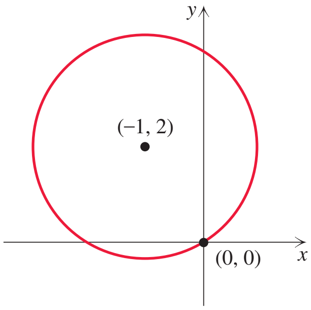
64. Center at $(-3, 9)$ and passing through the origin

65. Center at $(0, 0)$ and passing through $\left(\sqrt{2}/2, \sqrt{2}/2\right)$

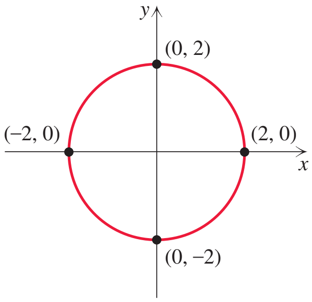
66. Center at $(0, 0)$ and passing through $\left(\sqrt{3}/2, 1/2\right)$

Write the standard equation for each of the following circles.

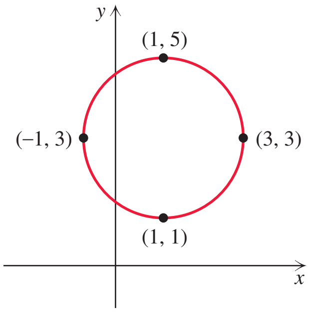
67.



68.



69.



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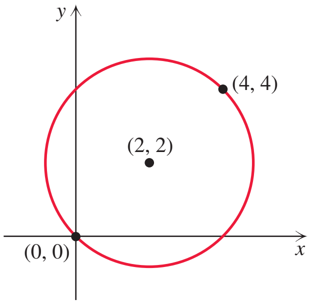


Find all real numbers a such that the given point is on the circle $x^2 + y^2 = 1$.

71. $(a, 3/5)$ **72.** $(a, -1/2)$

73. $(-2/5, a)$ **74.** $(2/3, a)$

Sketch the graph of each linear equation. Be sure to find and show the x- and y-intercepts.

75. $y = 3x - 4$ **76.** $y = 5x - 5$

77. $3x - y = 6$ **78.** $5x - 2y = 10$

79. $x + y = 80$ **80.** $2x + y = -100$

81. $x = 3y - 90$ **82.** $x = 80 - 2y$

83. $\frac{1}{2}x - \frac{1}{3}y = 600$ **84.** $\frac{2}{3}y - \frac{1}{2}x = 400$

85. $2x + 4y = 0.01$ **86.** $3x - 5y = 1.5$

Graph each equation in the rectangular coordinate system.

87. $x = 5$ **88.** $y = -2$ **89.** $y = 4$

90. $x = -3$ **91.** $x = -4$ **92.** $y = 5$

93. $y - 1 = 0$ **94.** $5 - x = 4$

Graph each of the following equations on a graphing calculator. Choose a viewing window that shows both the x- and y-intercepts, then draw the graph on paper with the x- and y-axes labeled appropriately.

95. $y = x - 20$ **96.** $y = 999x - 100$

97. $500x + y = 3000$ **98.** $200x - 300y = 1$

MODELING

Solve each problem.

99. A right triangle has legs with lengths 6 ft and 8 ft. What is the length of the hypotenuse?

100. A right triangle has a hypotenuse of 10 ft. If one leg is 4 ft, then what is the length of the other leg?

101. *Full Conduit* A conduit with an inside diameter of 4 cm can accommodate two round wires each having a diameter of 2 cm, as shown in the accompanying figure. Two smaller wires will also fit on the sides.

a. What is the diameter of each smaller wire?

b. Find the equations of the two smallest circles in the accompanying figure.

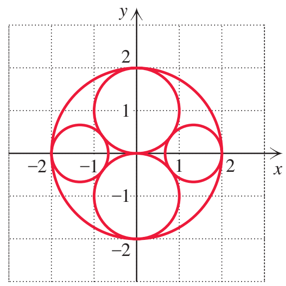


Figure for Exercise 101

102. *Perpendicular Tangents* A circle of radius 5 intersects a circle of radius 12 so that the tangent lines at the points of intersection are perpendicular. What is the distance between the centers of the circles?

103. *More Wires* A round conduit with an inside diameter of 4 cm can accommodate two circular wires each having a diameter of 2 cm, as shown in the accompanying figure. The smallest circle in the figure is tangent to the circle of radius 2, a circle of radius 1, and the x-axis. Find its equation.

104. *Three More* Three more circles could be positioned in the accompanying figure so that they are tangent to the circle of radius 2, a circle of radius 1, and the x-axis. Find their equations.

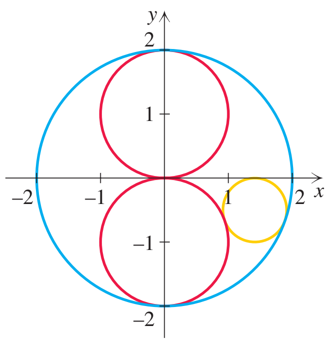


Figure for Exercises 103 and 104

105. *First Marriage* The median age at first marriage for women went from 20.8 in 1970 to 27.4 in 2018 as shown in the figure (Census Bureau, www.census.gov). Find the midpoint of the line segment in the figure and interpret your result.

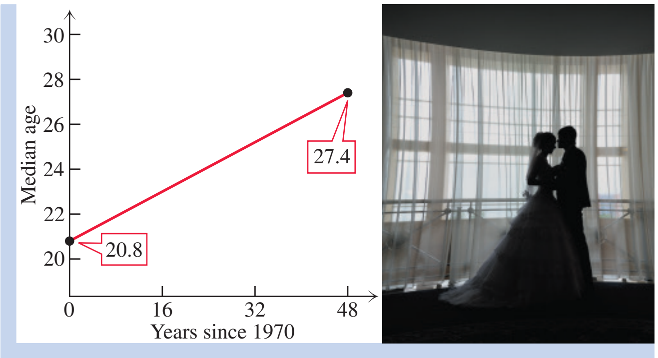


Figure for Exercise 105

106. *Unmarried Couples* The number of unmarried-couple households, h (in millions), can be approximated using the equation

$$h = 0.229n + 5.203,$$

where n is the number of years since 2000 (Census Bureau, www.census.gov).

a. Find and interpret the n-intercept for the line.

b. Find and interpret the h-intercept for the line.

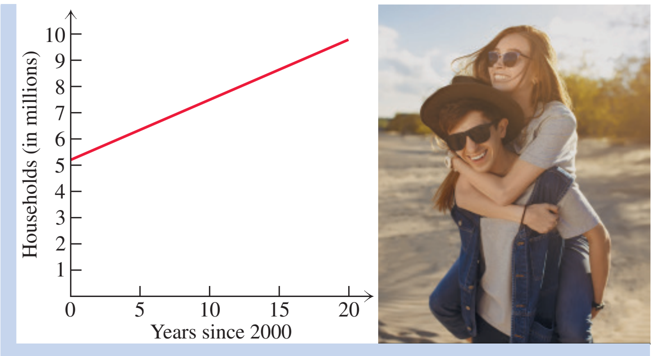


Figure for Exercise 106

WRITING/DISCUSSION

107. *Finding Points* Can you find two points such that their coordinates are integers and the distance between them is 10? $\sqrt{10}$? $\sqrt{19}$? Explain.

108. *Equidistant Midpoint* A right triangle has vertices (0, 0), (1, 0), and $(1, \sqrt{3})$. Find the midpoint of the hypotenuse. Find the distance from the midpoint of the hypotenuse to each vertex.

109. *Cooperative Learning* Working in small groups, plot the points $(-1, 3)$ and (4, 1) on a sheet of graph paper. Assuming that these two points are adjacent vertices of a square, find the other two vertices. Now select your own pair of adjacent vertices and "complete the square." Next, start with the points (x_1, y_1) and (x_2, y_2) as adjacent vertices of a square and write expressions for the coordinates of the other two vertices.

110. *Cooperative Learning* Repeat the previous exercise, assuming that the first two points are opposite vertices of a square.

OUTSIDE THE BOX

111. *Paying Up* A king agreed to pay his gardener one dollar's worth of titanium per day for seven days of work on the castle grounds. The king has a seven-dollar bar of titanium that is segmented so that it can be broken into seven one-dollar pieces, but it is bad luck to break a seven-dollar bar of titanium more than twice. How can the king make two breaks in the bar and pay the gardener exactly one dollar's worth of titanium per day for seven days?

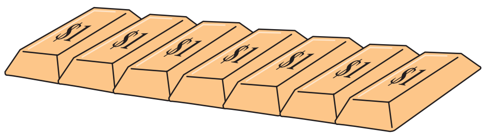


Figure for Exercise 111

112. *Midpoints* Find the exact area of the triangle in which $(3, -1)$, (4, 4) and (1, 2) are the midpoints of its three sides.

P.1 POP QUIZ

1. Find the distance between $(-1, 3)$ and (3, 5).
2. Find the center and radius for the circle $(x - 3)^2 + (y + 5)^2 = 81$.
3. Find the center and radius for the circle $x^2 + 4x + y^2 - 10y = -28$.
4. Find the equation of the circle that passes through the origin and has center at (3, 4).
5. Find all intercepts for $2x - 3y = 12$.
6. Which point is on both of the lines $x = 5$ and $y = -1$?

P.2 Functions

In Section P.1 we studied ordered pairs and graphed sets of ordered pairs. In this section we continue studying sets of ordered pairs, particularly those in which the second coordinate is determined by the first coordinate.

The Function Concept

If you spend \$10 on gasoline, then the price per gallon determines the number of gallons that you get. The number of hours that you sleep before a test might have an influence on but does not determine your grade on the test. If the value of a variable y is determined by the value of another variable x, then y **is a function of** x. The phrase "is a function of" means "is determined by." *If there is more than one value for y corresponding to a particular x-value, then y is not determined by x, and y is not a function of x.*

The value of one variable might be determined by the values of two or more other variables. For example, the volume of a right circular cylinder is a function of its radius and height $(V = \pi r^2 h)$. The volume of a rectangular box is a function of its length, width, and height $(V = LWH)$. The basic trigonometric functions that we will be studying in this text are all functions of one variable, but on occasion we will encounter functions of more than one variable.

EXAMPLE 1 Applying the function concept

Decide whether a is a function of b, b is a function of a, or neither.

a. Let a represent a positive integer smaller than 100 and b represent the number of divisors of a.
b. Let a represent the age of a U.S. citizen and b represent the number of days since his/her birth.
c. Let a represent the age of a U.S. citizen and b represent his/her annual income.
d. Let b represent any real number and a represent its square.

Solution

a. We can determine the number of divisors of any positive integer smaller than 100. So b is a function of a. We cannot determine the integer knowing the number of divisors because different integers have the same number of divisors. So a is not a function of b.
b. The number of days since a person's birth certainly determines the age of the person in the usual way. So a is a function of b. However, you cannot determine the number of days since a person's birth from the age. You need more information. For example, the number of days since the birth for two one-year-olds could be 370 or 380 days. So b is not a function of a.
c. We cannot determine the income from the age or the age from the income. We would need more information. Even though age and income are related, the relationship is not strong enough to say that either one is a function of the other.
d. Since every number has a unique square, the square is determined by the number. So a is a function of b. You cannot determine what the number is from knowing its square. For example, if you know the square is 4, then the number is either 2 or -2. So b is not a function of a.

TRY THIS. Let p be the price of a grocery item and t be the amount of sales tax at 5% on that item. Determine whether p is a function of t, t is a function of p, or neither.

The functions that we study will usually be defined by formulas. For example, if y is the square of x, then y is a function of x. We can write the formula or equation

$y = x^2$ and we say that $y = x^2$ is a function. Because y is determined by x, we say that y is the **dependent variable** and x is the **independent variable.** When we use the variables x and y, we always use x for the independent variable and y for the dependent variable.

A function provides a rule for pairing values of the independent variable with values of the dependent variable. The function $y = x^2$ pairs 3 with 9, 4 with 16, and so on. We write this pairing as (3, 9), (4, 16), and so on. Of course, there are infinitely many ordered pairs that satisfy $y = x^2$. This set of ordered pairs can be thought of as the function.

Definition: Function

A **function** is a set of ordered pairs in which no two pairs have the same first coordinate and different second coordinates.

For example, the set $\{(1, 1), (2, 4), (-2, 4)\}$ is a function because no two pairs have the same first coordinate and different second coordinates. The set $\{(2, 2), (9, 3), (9, -3)\}$ is not a function because (9, 3) and $(9, -3)$ have the same first coordinate and have different second coordinates. Note that we do not allow the same first coordinate with different second coordinates because we want the second coordinate to be determined by the first coordinate.

A calculator is a function machine. Built-in functions on a calculator are marked with symbols such as $\sqrt{x}$, x^2, $x!$, 10^x, e^x, $\ln(x)$, $\sin(x)$, $\cos(x)$, and so on. When you provide the first coordinate and use one of these functions, the calculator finds the appropriate second coordinate. The y-coordinate is certainly determined by the x-coordinate because the calculator will not produce two different y-coordinates corresponding to a given x-coordinate.

Constructing Functions

The ordered pairs in a function can be specified by a verbal description, list, table, formula, or graph. The most common and efficient way to specify the ordered pairs of a function is to give a formula that will produce them. For example, the function $A = \pi r^2$ determines the area of a circle when the radius is given and the function $y = \sqrt{x}$ determines the nonnegative square root of a nonnegative number. In the next example we find a formula for, or construct, a function.

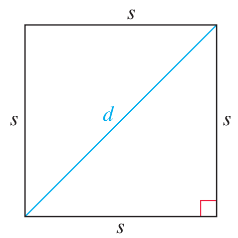


Figure P.16

EXAMPLE 2 Constructing a function

Given that a square has diagonal of length d and side of length s, write the area A as a function of the length of the diagonal.

Solution

The area of a square is given by $A = s^2$. The diagonal is the hypotenuse of a right triangle as shown in Fig. P.16. Apply the Pythagorean theorem to the right triangle in the figure:

$$\begin{aligned} s^2 + s^2 &= d^2 \\ 2s^2 &= d^2 \\ s^2 &= \frac{d^2}{2} \end{aligned}$$

Since $A = s^2$ and $s^2 = \frac{d^2}{2}$ we have $A = \frac{d^2}{2}$, which expresses the area of a square as a function of the length of its diagonal.

TRY THIS. A square has perimeter P and sides of length s. Write the side as a function of the perimeter.

Graphs of Functions

The graph of a function is a picture of all points whose ordered pairs make up the function. We can graph a function by calculating enough ordered pairs to determine the shape of the graph. When you graph functions, try to anticipate what the graph will look like, and after the graph is drawn, pause to reflect on the shape of the graph and the type of function that produced it. If you have a graphing calculator, use it to help you graph the equations in the following example. Remember that a graphing calculator shows only finitely many points and a graph consists of infinitely many points. After looking at the display of a graphing calculator, you must still decide what the entire graph looks like.

The **domain** of a function is the set of all first coordinates of the ordered pairs. The **range** of a function is the set of all second coordinates of the ordered pairs. If the domain is not specified, it is assumed to be all real numbers that can be used in the expression defining the function. The corresponding second coordinates must also be real numbers.

EXAMPLE 3 Graphing by plotting ordered pairs

Graph each function and state the domain and range.

a. $y = x^2$
b. $y = |x| - 40$

Solution

a. Make a table of ordered pairs that satisfy $y = x^2$:

x	-2	-1	0	1	2
$y = x^2$	4	1	0	1	4

These ordered pairs indicate a graph in the shape shown in Fig. P.17. This curve is called a **parabola.** The domain is $(-\infty, \infty)$ because any real number can be used for x in $y = x^2$. Since all y-coordinates are nonnegative, the graph is on or above the x-axis. The range is $[0, \infty)$.

The calculator graph in Fig. P.18 supports these conclusions. The calculator plots many more points than we are willing to plot by hand.

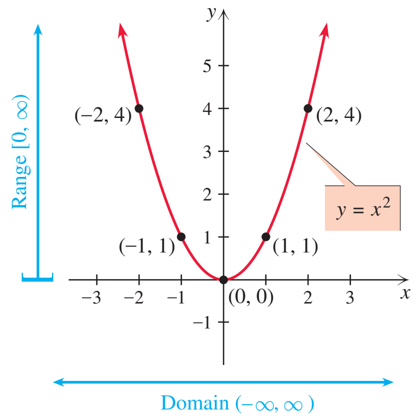


Figure P.17

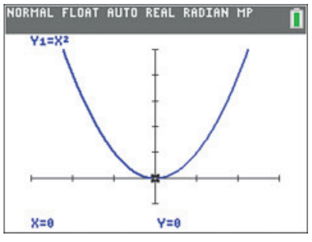


Figure P.18

b. Make a table of ordered pairs that satisfy $y = |x| - 40$:

x	-40	-20	0	20	40
$y = |x| - 40$	0	-20	-40	-20	0

Plotting these ordered pairs suggests the V-shaped graph of Fig. P.19 on the next page. From the graph we see that the domain is $(-\infty, \infty)$ and the range is $[-40, \infty)$.

The calculator graph in Fig. P.20 supports these conclusions.

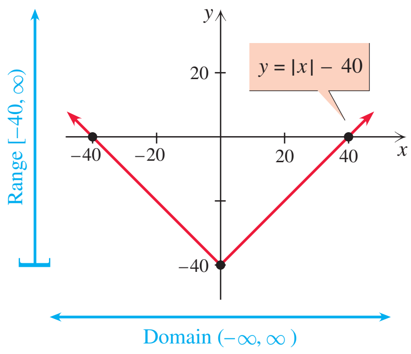


Figure P.19

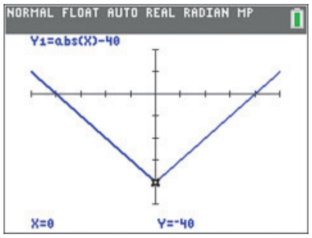


Figure P.20

TRY THIS. Graph $y = |x| + 2$ and state the domain and range.

The next example shows two functions whose domain is not the entire set of real numbers.

EXAMPLE 4 Graphing by plotting ordered pairs

Graph each function and state the domain and range.

a. $y = \sqrt{x - 20}$ **b.** $y = \dfrac{1}{x}$

Solution

a. Make a table of ordered pairs that satisfy $y = \sqrt{x - 20}$.

x	20	21	24	29	36
$y = \sqrt{x - 20}$	0	1	2	3	4

Plot these ordered pairs and sketch a curve through them as shown in Fig. P.21. This curve is half of a parabola. If $\sqrt{x - 20}$ is a real number, then $x - 20 \geq 0$, or $x \geq 20$. So the domain is $[20, \infty)$. On the graph we see that the y-coordinates range from 0 upward. So the range is the nonnegative real numbers, $[0, \infty)$. The calculator graph in Fig. P.22 supports these conclusions.

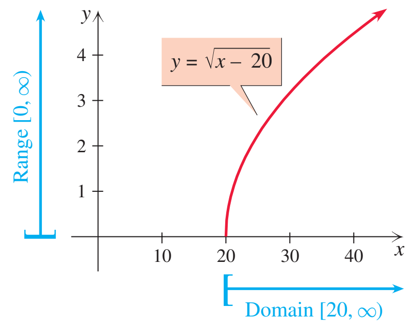


Figure P.21

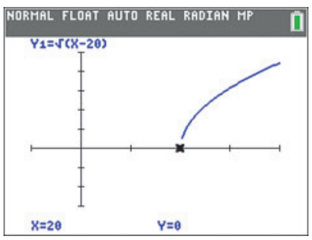


Figure P.22

b. Make a table of ordered pairs that satisfy $y = 1/x$:

x	-2	-1	$-1/2$	$1/2$	1	2
$y = 1/x$	$-1/2$	-1	-2	2	1	$1/2$

Note that as x gets larger, y gets smaller, and vice versa. Plotting these ordered pairs suggests the graph shown in Fig. P.23 on the next page. The domain is all real numbers except 0, $(-\infty, 0) \cup (0, \infty)$, and the range is also $(-\infty, 0) \cup (0, \infty)$. The calculator graph in Fig. P.24 supports these conclusions.

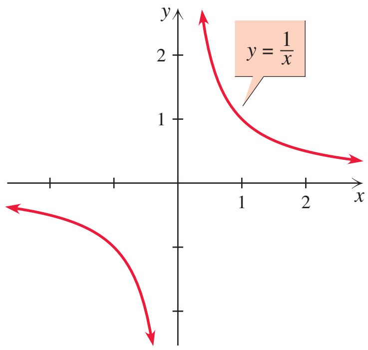


Figure P.23

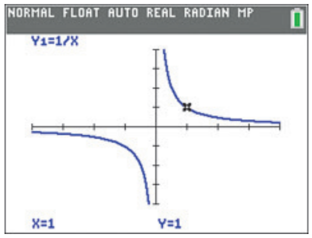


Figure P.24

TRY THIS. Graph $y = \sqrt{x + 3}$ and state the domain and range.

Function Notation

If y is a function of x, we may use the expression $f(x)$, rather than y, for the dependent variable. The notation $f(x)$ is read "the value of f at x" or simply "f of x." So y and $f(x)$ are both symbols for the second coordinate when the first coordinate is x, and we may write $y = f(x)$. For example, we could write $f(x) = \sqrt{x - 40}$ for the function $y = \sqrt{x - 40}$. The notation $f(x)$ is called **function notation.**

EXAMPLE 5 Using function notation

Find each of the following for the function $f(x) = \sqrt{2x + 9}$.

a. $f(20)$ **b.** x, if $f(x) = 0$ **c.** $f(-6)$ **d.** $f(-x)$

Solution

a. The expression $f(20)$ is the second coordinate when the first coordinate is 20. To find the value of $f(20)$ replace x by 20 in the formula $f(x) = \sqrt{2x + 9}$:

$$f(20) = \sqrt{2 \cdot 20 + 9} = \sqrt{49} = 7$$

b. To find a value of x so that the second coordinate is 0, we must solve $\sqrt{2x + 9} = 0$:

$$\begin{aligned} \sqrt{2x + 9} &= 0 \\ 2x + 9 &= 0 \\ 2x &= -9 \\ x &= -\frac{9}{2} \end{aligned}$$

So $x = -9/2$.

c. The expression $f(-6)$ is undefined, because the domain of $f(x) = \sqrt{2x + 9}$ is the interval $[-9/2, \infty)$.

d. The expression $f(-x)$ is the second coordinate when the first coordinate is $-x$. So replace x by $-x$ in the formula $f(x) = \sqrt{2x + 9}$:

$$f(-x) = \sqrt{2(-x) + 9} = \sqrt{-2x + 9}$$

TRY THIS. Let $f(x) = \sqrt{x + 1}$ **a.** Find $f(8)$. **b.** Find x if $f(x) = 5$.

We may use letters other than f in function notation. For example, we could have $h(x) = \sqrt{x}$ and $g(x) = x^2$. If a function describes some real application, then a letter that fits the situation is usually used. For example, if watermelons are \$3.00 each, then the cost of n watermelons, in dollars, is given by the function $C(n) = 3n$. The cost of 5 watermelons is $C(5) = 3 \cdot 5 = \$15$. In trigonometry the abbreviations sin, cos, and tan are used rather than a single letter to name the trigonometric functions. The dependent variables are written as $\sin(x)$, $\cos(x)$, and $\tan(x)$.

FOR THOUGHT... True or False? Explain.

1. The number of gallons of gas that can be purchased for \$20 is a function of the price per gallon.
2. Each student's exam grade is a function of the student's IQ.
3. Any set of ordered pairs is a function.
4. If $y = x^2$, then y is a function of x.
5. If x is the independent variable, then $f(x)$ is the dependent variable.
6. The domain of $f(x) = 1/x$ is $(-\infty, 0) \cup (0, \infty)$.
7. The domain of $g(x) = |x - 3|$ is $[3, \infty)$.
8. The range of $y = 8 - x^2$ is $(-\infty, 8]$.
9. If $f(t) = \frac{t-2}{t+2}$, then $f(0) = -1$.
10. If $f(x) = x - 5$ and $f(a) = 0$, then $a = 5$.

P.2 EXERCISES

CONCEPTS

Fill in the blank.

1. A set of ordered pairs in which no two have the same first coordinate and different second coordinates is a(n) __________
2. For a set of ordered pairs, the variable corresponding to the first coordinate is the __________ variable and the variable corresponding to the second coordinate is the __________ variable.
3. For a set of ordered pairs, the set of all first coordinates is the __________ and the set of all second coordinates is the __________.
4. The graph of $y = x^2$ is a(n) __________.
5. If the value of y is determined by the value of x, then y is a(n) __________ of x.
6. We write $y = x^2$ in __________ notation as $f(x) = x^2$.

SKILLS

For each pair of variables determine whether a is a function of b, b is a function of a, or neither.

7. a is the radius of any U.S. coin and b is its circumference.
8. a is the length of any rectangle with width 5 in. and b is its perimeter.
9. a is the length of any piece of U.S. paper currency and b is its denomination.
10. a is the diameter of any U.S. coin and b is its value.
11. a is the universal product code for any item at Walmart and b is its price.
12. a is the final exam score for any student in your class and b is the student's semester grade.
13. a is the time spent studying for the final exam for any student in your class and b is the student's final exam score.
14. a is the age of any adult male and b is his shoe size.
15. a is the height of any car in inches and b is its height in centimeters.
16. a is the cost of mailing any first-class letter and b is its weight.
17. a is any real number and b is the cube of that number.
18. a is any real number and b is the fourth power of that number.
19. a is any real number and b is the absolute value of that number.
20. a is any nonnegative real number and b is a square root of that number.

Consider a square with side of length s, diagonal of length d, perimeter P, and area A. Make a sketch.

21. Write A as a function of s.
22. Write s as a function of A.
23. Write s as a function of d.
24. Write d as a function of s.
25. Write P as a function of s.
26. Write s as a function of P.
27. Write A as a function of P.
28. Write d as a function of A.

Graph each function by plotting points and state the domain and range. If you have a graphing calculator, use it to check your results.

29. $y = 2x - 1$

30. $y = -x + 3$

31. $y = 5$

32. $y = -4$

33. $y = x^2 - 20$

34. $y = x^2 + 50$

35. $y = 40 - x^2$

36. $y = -10 - x^2$

37. $y = x^3$

38. $y = -x^3$

39. $y = \sqrt{x - 10}$

40. $y = \sqrt{x + 30}$

41. $y = \sqrt{x} + 30$

42. $y = \sqrt{x} - 50$

43. $y = |x| - 40$

44. $y = 2|x|$

45. $y = |x - 20|$

46. $y = |x + 30|$

Let $f(x) = 3x^2 - x$, $g(x) = 4x - 2$, *and* $k(x) = |x + 3|$. *Find the following.*

47. $f(2)$

48. $f(-4)$

49. $g(-1)$

50. $g(-2)$

51. $k(5)$

52. $k(-4)$

53. $f(-1) + g(-1)$

54. $f(3) \cdot k(3)$

55. $f(-5) - k(-5)$

56. $\dfrac{f(1)}{g(1)}$

57. $f(a)$

58. $g(b)$

59. $f(-x)$

60. $g(-x)$

61. x, if $f(x) = 0$

62. x, if $g(x) = 3$

63. a, if $k(a) = 4$

64. t, if $f(t) = 10$

MODELING

Solve each problem.

65. *Aerobics* If a woman in an aerobic dance class burns 353 calories per hour, express the number of calories burned, C, as a function of the number of hours danced, n.

66. *Paper Consumption* If the average American uses 580 lb of paper annually, express the total annual paper consumption of the United States, P, as a function of the size of the population, n.

67. *Cost of Window Cleaning* If a window cleaner charges \$50 per visit plus \$35 per hour, express the total charge C as a function of the number of hours worked, n.

68. *Cost of a Sundae* At Doubledipski's ice cream parlor, the cost of a sundae is \$2.50 plus \$0.50 for each topping. Express the total cost of a sundae, C, as a function of the number of toppings, n.

69. *Capsize Control* The capsize screening value C is an indicator of a sailboat's suitability for extended offshore sailing. C is determined by the function $C = 4D^{-1/3}B$, where D is the displacement of the boat in pounds and B is its beam (or width) in feet. Sketch the graph of this function for B ranging from 0 to 20 ft, assuming that D is fixed at 22,800 lb. Find C for the Island Packet 40, which has a displacement of 22,800 lb and a beam of 12 ft 11 in. (Island Packet Yachts, www.ipy.com).

Figure for Exercises 69 and 70

70. *Limiting the Beam* The International Offshore Rules require that the capsize screening value C (from the previous exercise) be less than or equal to 2 for safety. What is the maximum allowable beam (to the nearest inch) for a boat with a displacement of 22,800 lb? For a fixed displacement, is a boat more or less likely to capsize as its beam gets larger?

71. *Bore and Stroke* The displacement D of an engine is given by the function

$$D = \frac{\pi}{4}B^2 \cdot S \cdot N,$$

where B is the bore (the diameter of a cylinder), S is the stroke (distance the piston moves), and N is the number of cylinders. The two-cylinder Harley-Davidson Evo engine has a bore of 3.498 in. and a stroke of 4.250 in. Find its displacement.

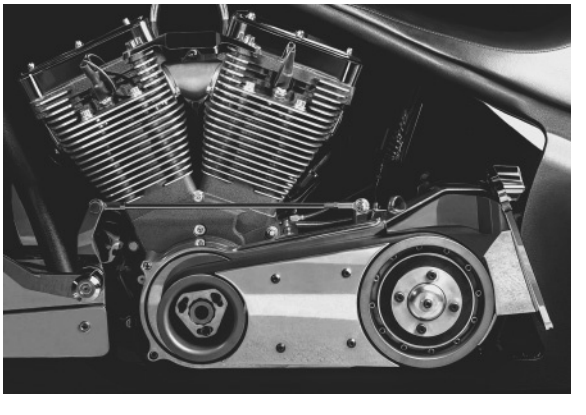


Figure for Exercise 71

72. *Rebuilt Engine* If the bore of the Evo engine from the last exercise is increased by 0.020 in., then what is the increase in displacement?

73. *Finding the Bore* Find a formula that expresses the bore B as a function of the displacement D, stroke S, and number of cylinders N. See Exercise 71.

74. *Compression Ratio* The compression ratio CR for an engine is a function of the bore B, the stroke S, and the clearance volume V, where

$$CR = 1 + \frac{\pi B^2 \cdot S}{4V}.$$

The clearance volume is the volume of air in the cylinder when the piston is at top dead center. Find a formula that expresses the clearance volume V as a function of CR, B, and S.

REVIEW

75. According to the __________ theorem, the sum of the squares of the __________ of a right triangle is equal to the square of the __________.

76. The equation $(x - 3)^2 + (x + 2)^2 = 36$ is the equation of a(n) __________ with __________ 6 and __________ $(3, -2)$.

77. Find the distance between the points $(2, -4)$ and $(-3, -6)$.

78. Find the midpoint of the line segment whose endpoints are $(4, -8)$ and $(-6, 16)$.

79. Find the x- and y-intercepts for the line $4x - 6y = 40$.

80. The sides of a rectangle are 3 feet and 7 feet. Find the length of the diagonal.

OUTSIDE THE BOX

81. *Powers of Three* What is x if

$$\frac{1}{27} \cdot 3^{100} \cdot \frac{1}{81} \cdot 9^x = \frac{1}{3} \cdot 3^x?$$

82. *Sum of Cubes* If $a + b = 3$ and $a^2 + b^2 = 89$, then what is the value of $a^3 + b^3$?

P.2 POP QUIZ

1. Is the radius of a circle a function of its area?

2. Express the side of a square s as a function of its area A.

3. Let a be a positive real number and b be a real number such that $b^2 = a$. Is b a function of a?

4. What is the domain of $y = \sqrt{x - 1}$?

5. What is the range of $y = x^2 + 2$?

6. What is $f(3)$ if $f(x) = 3x + 6$?

7. Find a if $f(a) = 10$ and $f(x) = 2x - 4$.

P.3 Families of Functions, Transformations, and Symmetry

As seen in Section P.2, different functions have similar graphs. In this section we will see how those functions are related. If a, h, and k are real numbers with $a \neq 0$, then $y = af(x - h) + k$ is a **tranformation** $y = f(x)$. All of the transformations of a function form a **family of functions.** The quadratic or square family has the form $y = a(x - h)^2 + k$ and the graphs are **parabolas.** The square-root family (halves of parabolas) has the form $y = a\sqrt{x - h} + k$. The absolute value family (V-shaped graphs) has the form $y = a|x - h| + k$. In Chapter 2 we will use these ideas to study families of trigonometric functions.

Horizontal Translation

If you can move a graph to the left or right (without changing its shape) so that it coincides with another graph, then the graphs are *horizontal translations* of each other. This geometric idea is made more precise using algebra in the following definition.

Definition: Translation to the Right or Left

If $h > 0$, then the graph of $y = f(x - h)$ is a **translation of h units to the right** of the graph of $y = f(x)$. If $h < 0$, then the graph of $y = f(x - h)$ is a **translation of $|h|$ units to the left** of the graph of $y = f(x)$.

According to the order of operations, the first operation to perform in the formula $y = af(x - h) + k$ is to subtract h from x. Then $f(x - h)$ is multiplied by a, and finally k is added on. The order is important here, and we will study the effects of these numbers in the order h, a, and k.

EXAMPLE 1 Translations to the right or left

Graph $f(x) = \sqrt{x}$, $g(x) = \sqrt{x - 3}$, and $h(x) = \sqrt{x + 5}$ on the same coordinate plane.

Solution

First sketch $f(x) = \sqrt{x}$ through (0, 0) (1, 1), and (4, 2) as shown in Fig. P.25. Since the first operation of g is to subtract 3, we get the corresponding points by adding 3 to each x-coordinate. So g goes through (3, 0), (4, 1), and (7, 2). Since the first operation of h is to add 5, we get corresponding points by subtracting 5 from the x-coordinates. So h goes through $(-5, 0)$, $(-4, 1)$, and $(-1, 2)$.

The calculator graphs of f, g, and h are shown in Fig. P.26. Be sure to note the difference between $y = \sqrt{\ }(x) - 3$ and $y = \sqrt{\ }(x - 3)$ on a calculator.

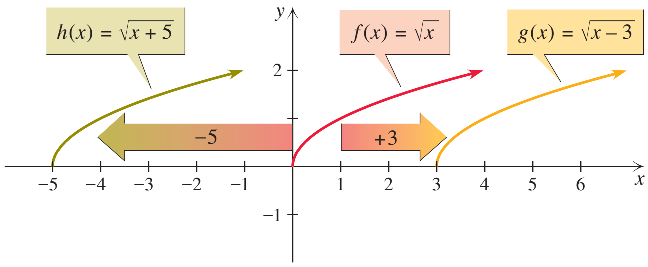


Figure P.25

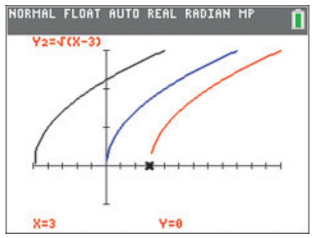


Figure P.26

TRY THIS. Graph $f(x) = \sqrt{x}$, $g(x) = \sqrt{x - 2}$, and $h(x) = \sqrt{x + 1}$ on the same coordinate plane.

Notice that $y = \sqrt{x - 3}$ lies 3 units to the *right* and $y = \sqrt{x + 5}$ lies 5 units to the *left* of $y = \sqrt{x}$. The next example shows two more horizontal translations.

EXAMPLE 2 Horizontal translations

Sketch the graph of each function.

a. $f(x) = |x - 1|$
b. $f(x) = (x + 3)^2$

Solution

a. The function $f(x) = |x - 1|$ is in the absolute-value family and its graph is a translation one unit to the right of $g(x) = |x|$. Calculate a few ordered pairs to get an accurate graph. The points (0, 1), (1, 0), and (2, 1) are on the graph of $f(x) = |x - 1|$ shown in Fig. P.27.

b. The function $f(x) = (x + 3)^2$ is in the square family and its graph is a translation three units to the left of the graph of $g(x) = x^2$. Calculate a few ordered pairs to get an accurate graph. The points $(-3, 0)$, $(-2, 1)$, and $(-4, 1)$ are on the graph shown in Fig. P.28.

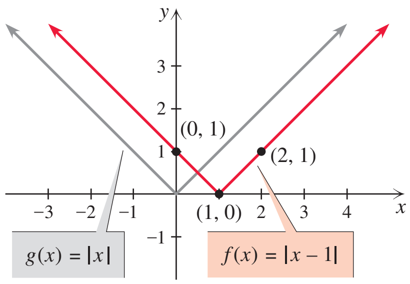


Figure P.27

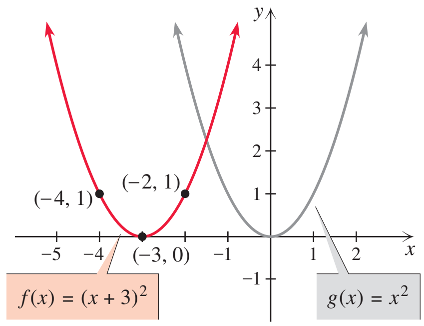


Figure P.28

TRY THIS. Graph $f(x) = (x - 2)^2$.

Reflection

If a graph is a mirror image of another, then the graphs are *reflections* of each other. As with horizontal translation, this idea is also made precise by using algebra.

Definition: Reflection

The graph of $y = -f(x)$ is a **reflection** in the x-axis of the graph of $y = f(x)$.

In the general formula $y = af(x - h) + k$, a reflection is accomplished by choosing $a = -1$ and $h = k = 0$. All points on the graph of $y = -f(x)$ can be obtained by simply changing the signs of all of the y-coordinates of the points on the graph of $y = f(x)$.

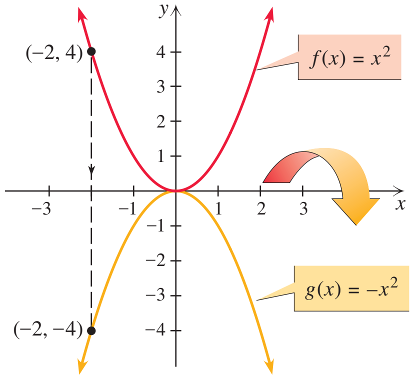


Figure P.29

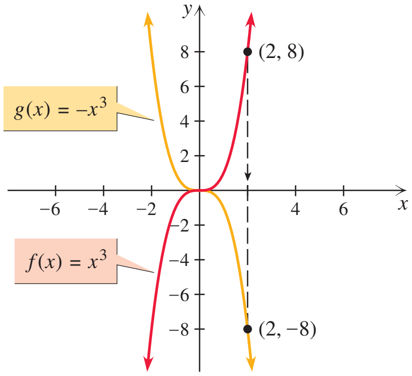


Figure P.30

EXAMPLE 3 **Graphing using reflection**

Graph each pair of functions on the same coordinate plane.

a. $f(x) = x^2$, $g(x) = -x^2$
b. $f(x) = x^3$, $g(x) = -x^3$
c. $f(x) = |x|$, $g(x) = -|x|$

Solution

a. The graph of $f(x) = x^2$ goes through $(0, 0)$, $(\pm 1, 1)$, and $(\pm 2, 4)$. The graph of $g(x) = -x^2$ goes through $(0, 0)$, $(\pm 1, -1)$, and $(\pm 2, -4)$ as shown in Fig. P.29.

b. Make a table of ordered pairs for f as follows:

x	-2	-1	0	1	2
$f(x) = x^3$	-8	-1	0	1	8

Sketch the graph of f through these ordered pairs as shown in Fig. P.30. Since $g(x) = -f(x)$, the graph of g can be obtained by reflecting the graph of f in the x-axis. Each point on the graph of f corresponds to a point on the graph of g with the opposite y-coordinate. For example, $(2, 8)$ on the graph of f corresponds to $(2, -8)$ on the graph of g. Both graphs are shown in Fig. P.30.

c. The graph of f is the familiar V-shaped graph of the absolute value function as shown in Fig. P.31 on the next page. Since $g(x) = -f(x)$, the graph of g can be obtained by reflecting the graph of f in the x-axis. Each point on the graph of f corresponds to a point on the graph of g with the opposite y-coordinate. For example, $(2, 2)$ on f corresponds to $(2, -2)$ on g. Both graphs are shown in Fig. P.31 on the next page.

TRY THIS. Graph $f(x) = \sqrt{x}$ and $g(x) = -\sqrt{x}$ on the same coordinate plane.

We do not include reflections in the y-axis in our function families, but it is interesting to note that you get a reflection in the y-axis simply by replacing x with $-x$ in the formula for the function. For example, graph $y_1 = \sqrt{x}$ and $y_2 = \sqrt{-x}$ on your graphing calculator.

Stretching and Shrinking

Two graphs that have similar shapes may be related by a vertical *stretching* or *shrinking* of one into the other. This geometric idea is defined precisely using algebra as follows.

Definitions: Stretching and Shrinking

The graph of $y = af(x)$ is obtained from the graph of $y = f(x)$ by

1. **stretching** the graph of $y = f(x)$ by a when $a > 1$, or
2. **shrinking** the graph of $y = f(x)$ by a when $0 < a < 1$.

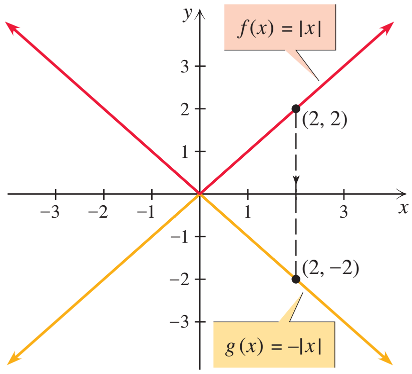


Figure P.31

Note that stretching and shrinking are defined for positive values of a. If a is negative, then reflection occurs along with stretching or shrinking.

EXAMPLE 4 Graphing using stretching and shrinking

In each case graph the three functions on the same coordinate plane.

a. $f(x) = \sqrt{x}$, $g(x) = 2\sqrt{x}$, $h(x) = \dfrac{1}{2}\sqrt{x}$

b. $f(x) = x^2$, $g(x) = 2x^2$, $h(x) = \dfrac{1}{2}x^2$

Solution

a. The graph of $f(x) = \sqrt{x}$ goes through (0, 0), (1, 1), and (4, 2) as shown in Fig. P.32. The graph of g is obtained by stretching the graph of f by a factor of 2. So g goes through (0, 0), (1, 2), and (4, 4). The graph of h is obtained by shrinking the graph of f by a factor of $\frac{1}{2}$. So h goes through (0, 0), $\left(1, \frac{1}{2}\right)$, and (4, 1).

The functions f, g, and h are shown on a graphing calculator in Fig. P.33. Note how the viewing window affects the shape of the graph. The curves do not appear as separated on the calculator as they do in Fig. P.32.

b. The graph of $f(x) = x^2$ is the familiar parabola shown in Fig. P.34. We stretch it by a factor of 2 to get the graph of g and shrink it by a factor of $\frac{1}{2}$ to get the graph of h.

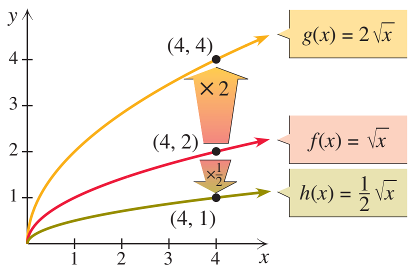


Figure P.32

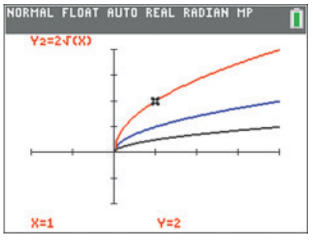


Figure P.33

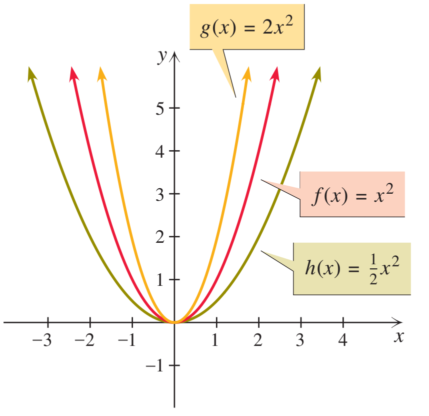


Figure P.34

TRY THIS. Graph $f(x) = |x|$, $g(x) = 3|x|$, and $h(x) = \frac{1}{3}|x|$ on the same coordinate plane.

Note that the graph of $y = 2x^2$ has exactly the same shape as the graph of $y = x^2$ if we simply change the scale on the y-axis as in Fig. P.35.

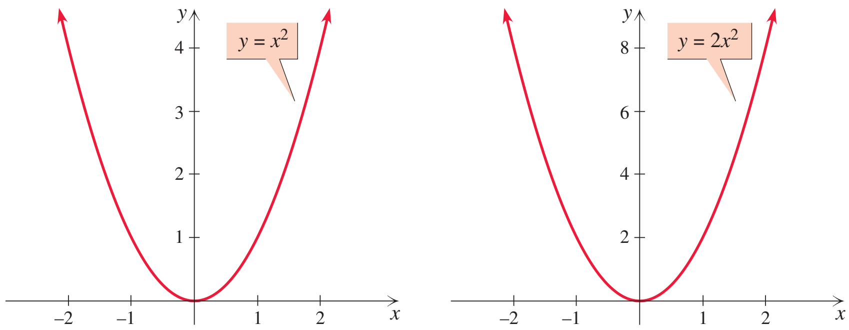


Figure P.35

Vertical Translation

In the formula $y = af(x - h) + k$, the last operation is addition of k. If we add the constant k to all y-coordinates on the graph of $y = f(x)$, the graph will be translated up or down depending on whether k is positive or negative.

Definition: Translation Upward or Downward

If $k > 0$, then the graph of $y = f(x) + k$ is a **translation of k units upward** of the graph of $y = f(x)$. If $k < 0$, then the graph of $y = f(x) + k$ is a **translation of $|k|$ units downward** of the graph of $y = f(x)$.

EXAMPLE 5 Translations upward or downward

Graph the three given functions on the same coordinate plane.

a. $f(x) = \sqrt{x}$, $g(x) = \sqrt{x} + 3$, $h(x) = \sqrt{x} - 5$
b. $f(x) = x^2$, $g(x) = x^2 + 2$, $h(x) = x^2 - 3$

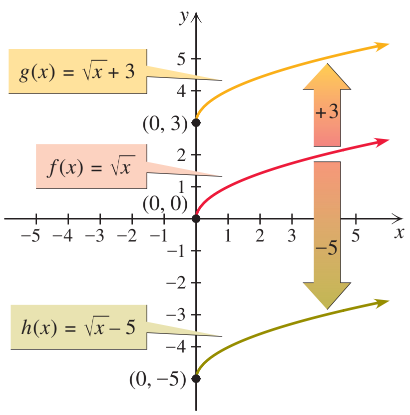


Figure P.36

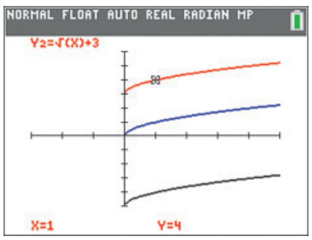


Figure P.37

Solution

a. First sketch $f(x) = \sqrt{x}$ through (0, 0), (1, 1), and (4, 2) as shown in Fig. P.36. Since $g(x) = \sqrt{x} + 3$ we can add 3 to the y-coordinate of each point to get (0, 3), (1, 4), and (4, 5). Sketch g through these points. Every point on f can be moved up 3 units to obtain a corresponding point on g. We now subtract 5 from the y-coordinates on f to obtain points on h. So h goes through $(0, -5)$, $(1, -4)$, and $(4, -3)$.

The relationship between f, g, and h can be seen with a graphing calculator in Fig. P.37. You should experiment with your graphing calculator to see how a change in the formula changes the graph.

b. First sketch the familiar graph of $f(x) = x^2$ through $(\pm 2, 4)$, $(\pm 1, 1)$, and (0, 0) as shown in Fig. P.38. Since $g(x) = f(x) + 2$, the graph of g can be obtained by translating the graph of f upward two units. Since $h(x) = f(x) - 3$, the graph of h can be obtained by translating the graph of f downward three units. For example, the point $(-1, 1)$ on the graph of f moves up to $(-1, 3)$ on the graph of g and down to $(-1, -2)$ on the graph of h as shown in Fig. P.38 on the next page.

TRY THIS. Graph $f(x) = \sqrt{x}$, $g(x) = \sqrt{x} + 1$, and $h(x) = \sqrt{x} - 2$ on the same coordinate plane.

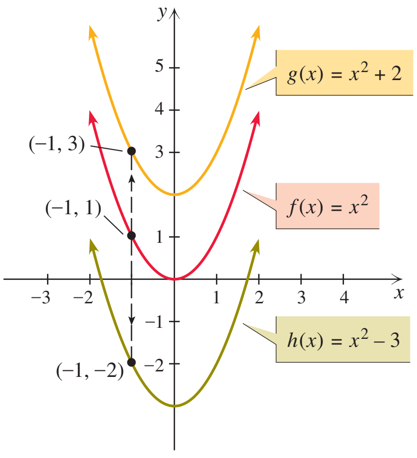


Figure P.38

Multiple Transformations

Any combination of stretching, shrinking, reflecting, horizontal translation, and vertical translation transforms one function into a new function. If a transformation does not change the shape of a graph, then it is a **rigid** transformation. If the shape changes, then it is **nonrigid.** Stretching and shrinking are nonrigid transformations. Translating (horizontally or vertically) and reflection are rigid transformations.

When graphing a function involving more than one transformation, $y = af(x - h) + k$, apply the transformations in the order that we discussed them: *h-a-k*. Remember that this order is simply the order of operations.

PROCEDURE

Multiple Transformations

To graph $y = af(x - h) + k$ apply the transformations in the following order:

1. Horizontal translation (h)
2. Reflecting/stretching/shrinking (a)
3. Vertical translation (k)

Note that the order in which you reflect and stretch or reflect and shrink does not matter. It does matter that you do vertical translation last. For example, if $y = x^2$ is reflected in the x-axis and then translated up one unit, the equation for the graph is $y = -x^2 + 1$. If $y = x^2$ is translated up one unit and then reflected in the x-axis, the equation for the graph in the final position is $y = -(x^2 + 1)$ or $y = -x^2 - 1$. Changing the order has resulted in different functions.

EXAMPLE 6 Graphing using several transformations

Use transformations to graph each function.

a. $y = -2(x - 3)^2 + 4$

b. $y = 4 - 2\sqrt{x + 1}$

Solution

a. This function is in the square family. So the graph of $y = x^2$ is translated 3 units to the right, reflected and stretched by a factor of 2, and finally translated 4 units upward. See Fig. P.39.

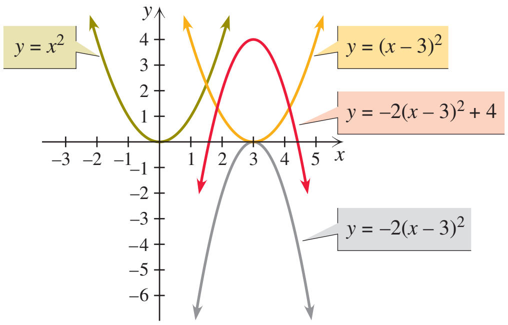


Figure P.39

b. Rewrite the function as $y = -2\sqrt{x + 1} + 4$ and recognize that it is in the square-root family. So we start with the graph of $y = \sqrt{x}$. The graph of $y = \sqrt{x + 1}$ is a horizontal translation one unit to the left of $y = \sqrt{x}$. Stretch by a factor of

2 to get $y = 2\sqrt{x + 1}$. Reflect in the x-axis to get $y = -2\sqrt{x + 1}$. Finally, translate 4 units upward to get the graph of $y = -2\sqrt{x + 1} + 4$. All of these graphs are shown in Fig. P.40.

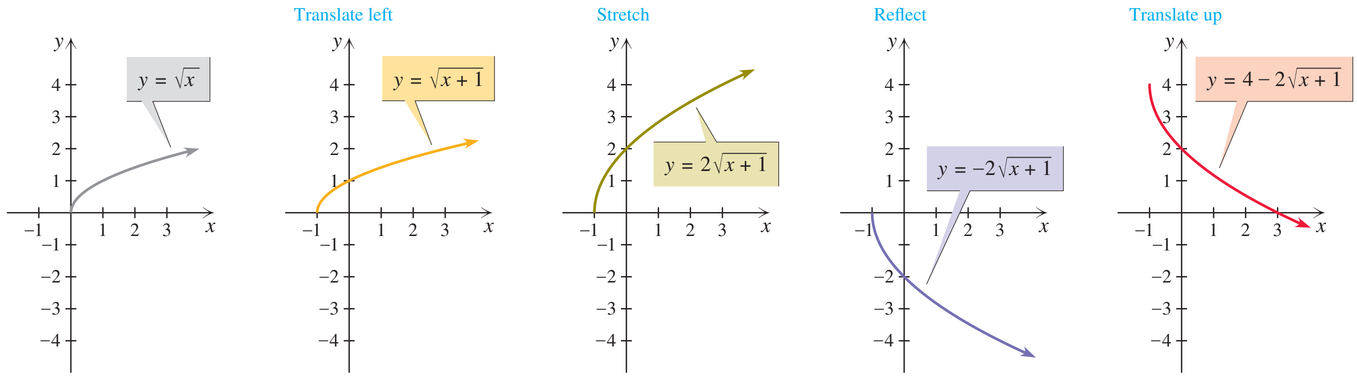


Figure P.40

TRY THIS. Graph $y = 4 - 2\,|x + 1|$.

Symmetry

The graph of $g(x) = -x^2$ is a reflection in the x-axis of the graph of $f(x) = x^2$. If the paper were folded along the x-axis, the graphs would coincide. See Fig. P.41. The symmetry that we call reflection occurs between two functions, but the graph of $f(x) = x^2$ has a symmetry within itself. Points such as $(2, 4)$ and $(-2, 4)$ are on the graph and are equidistant from the y-axis. Folding the paper along the y-axis brings all such pairs of points together. See Fig. P.42. The reason for this symmetry about the y-axis is the fact that $f(-x) = f(x)$ for any real number x. We get the same y-coordinate whether we evaluate the function at a number or at its opposite.

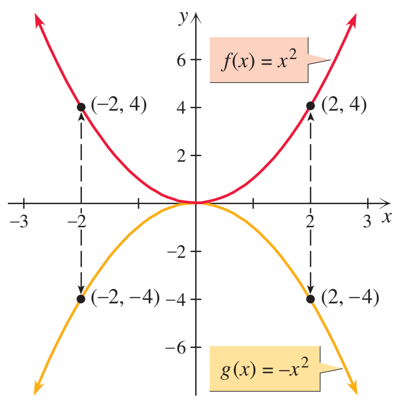


Figure P.41

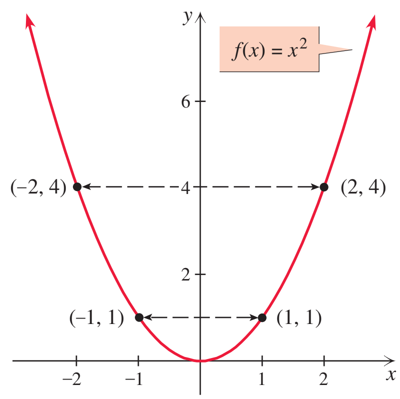


Figure P.42

Definition: Symmetric about the *y*-Axis

If $f(-x) = f(x)$ for every value of x in the domain of the function f, then f is an **even function** and its graph is **symmetric about the *y*-axis.**

The graph of $y = f(-x)$ is a reflection in the y-axis of the graph of $y = f(x)$. If the function is even, these graphs are identical. For an example, graph $y = x^2$ and $y = (-x)^2$. If the function is not even, the graphs are not identical, but each is still a reflection in the y-axis of the other. For an example, graph $y = x^3$ and $y = (-x)^3$.

Consider the graph of $f(x) = x^3$ shown in Fig. P.43. It is not symmetric about the y-axis like the graph of $f(x) = x^2$, but it has another kind of symmetry. On the graph of $f(x) = x^3$ we find pairs of points such as (2, 8) and $(-2, -8)$. These points are equidistant from the origin and on opposite sides of the origin. So the symmetry of this graph is about the origin. In this case, $f(x)$ and $f(-x)$ are not equal, but $f(-x) = -f(x)$.

Definition: Symmetric about the Origin

If $f(-x) = -f(x)$ for every value of x in the domain of the function f, then f is an **odd function** and its graph is **symmetric about the origin.**

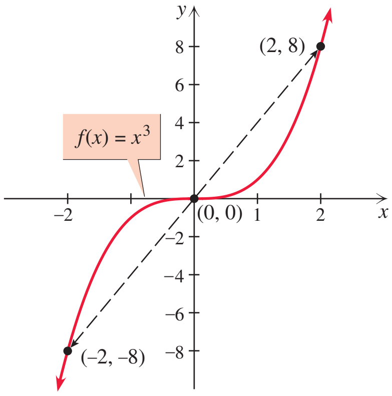


Figure P.43

We can look at a graph and see if it is symmetric about the y-axis or the origin, but it makes graphing easier and more accurate if we can identify symmetry *before* graphing. Using the definitions, we can determine whether a function is even, odd, or neither from the formula defining the function. If we know about symmetry (or the lack of it) in advance, then we know what to expect when we plot points.

EXAMPLE 7 Determining symmetry in a graph

Discuss the symmetry of the graph of each function.

a. $f(x) = 5x^3 - x$ **b.** $f(x) = |x| + 3$ **c.** $f(x) = x^2 - 3x + 6$

Solution

a. Replace x by $-x$ in the formula for $f(x)$ and simplify:

$$f(-x) = 5(-x)^3 - (-x) = -5x^3 + x$$

Is $f(-x)$ equal to $f(x)$ or the opposite of $f(x)$? Since $-f(x) = -5x^3 + x$, we have $f(-x) = -f(x)$. So f is an odd function and the graph is symmetric about the origin.

b. Since $|-x| = |x|$ for any real number x, we have $f(-x) = |-x| + 3 = |x| + 3$. Because $f(-x) = f(x)$, the function is even and the graph is symmetric about the y-axis.

c. In this case, $f(-x) = (-x)^2 - 3(-x) + 6 = x^2 + 3x + 6$. So $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. This function is neither odd nor even, and its graph has neither type of symmetry.

TRY THIS. Discuss the symmetry of the graph of $f(x) = -2x^2 + 5$.

FOR THOUGHT... True or False? Explain.

1. The graph of $f(x) = (-x)^4$ is a reflection in the x-axis of the graph of $g(x) = x^4$.
2. The graph of $f(x) = x^2 - 4$ lies 4 units to the right of the graph of $g(x) = x^2$.
3. The graph of $y = |x + 2| + 2$ is a translation 2 units to the right and 2 units upward of the graph of $y = |x|$.
4. The graph of $f(x) = -3$ is a reflection in the x-axis of the graph of $g(x) = 3$.
5. The functions $y = x^2 + 4x + 1$ and $y = (x + 2)^2 - 3$ have the same graph.
6. The graph of $y = -(x - 3)^2 - 4$ can be obtained by moving $y = x^2$ a distance of 3 units to the right and down 4 units, and then reflecting in the x-axis.
7. If $f(x) = -x^3 + 2x^2 - 3x + 5$, then $f(-x) = x^3 + 2x^2 + 3x + 5$.
8. The graphs of $f(x) = -\sqrt{x}$ and $g(x) = \sqrt{-x}$ are identical.
9. If $f(x) = x^3 - x$, then $f(-x) = -f(x)$.
10. The graph of $y = x^3$ is symmetric with respect to the origin.

P.3 EXERCISES

CONCEPTS

Fill in the blank.

1. Translating and reflecting are __________ transformations.
2. Stretching and shrinking are __________ transformations.
3. The graph of $y = -f(x)$ is a(n) __________ of the graph of $y = f(x)$.
4. The graph of $y = f(x) + k$ is a(n) __________ of the graph of $y = f(x)$ if $k > 0$ or a(n) __________ if $k < 0$.
5. The graph of $y = f(x - h)$ is a translation to the __________ of the graph of $y = f(x)$ if $h > 0$ or a translation to the __________ if $h < 0$.
6. The graph of $y = af(x)$ is obtained by __________ the graph of $y = f(x)$ if $a > 1$ or __________ the graph of $y = f(x)$ if $0 < a < 1$.
7. If $f(-x) = -f(x)$ for every x in the domain of f, then f is a(n) __________ function.
8. If $f(-x) = f(x)$ for every x in the domain of f, then f is a(n) __________ function.
9. The graph of $y = af(x - h) + k$ is a(n) __________ of the graph of $y = f(x)$.
10. All transformations of a function form a(n) __________ of functions.

SKILLS

Sketch the graphs of each pair of functions on the same coordinate plane.

11. $f(x) = \sqrt{x}, g(x) = -\sqrt{x}$
12. $f(x) = x^2 + 1, g(x) = -x^2 - 1$
13. $y = x, y = -x$
14. $y = \sqrt{4 - x^2}, y = -\sqrt{4 - x^2}$
15. $f(x) = |x|, g(x) = |x| - 4$
16. $f(x) = \sqrt{x}, g(x) = \sqrt{x} + 3$
17. $f(x) = x, g(x) = x + 3$
18. $f(x) = x^2, g(x) = x^2 - 5$
19. $y = x^2, y = (x - 3)^2$
20. $y = |x|, y = |x + 2|$
21. $f(x) = x^3, g(x) = (x + 1)^3$
22. $f(x) = \sqrt{x}, g(x) = \sqrt{x - 3}$
23. $y = \sqrt{x}, y = 3\sqrt{x}$
24. $y = |x|, y = \frac{1}{3}|x|$
25. $y = x^2, y = \frac{1}{4}x^2$
26. $y = x^2, y = 4 - x^2$

The figure for Exercises 27–34 shows eight parabolas. Match each function with its graph (a)–(h).

27. $y = x^2$
28. $y = (x - 4)^2 + 2$
29. $y = (x + 4)^2 - 2$
30. $y = -2(x - 2)^2$
31. $y = -2(x + 2)^2$
32. $y = -\frac{1}{2}x^2 - 4$
33. $y = \frac{1}{2}(x + 4)^2 + 2$
34. $y = -2(x - 4)^2 - 2$

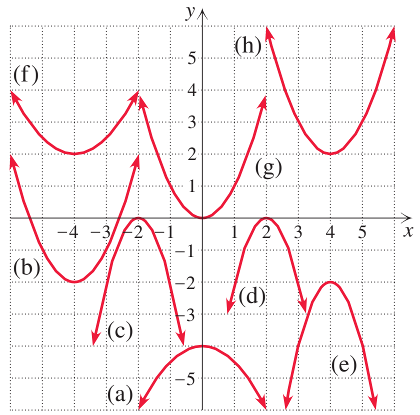


Figure for Exercises 27–34

Write the equation of each graph in its final position.

35. The graph of $y = x^2$ is translated 10 units to the right and 4 units upward.
36. The graph of $y = \sqrt{x}$ is translated 5 units to the left and 12 units downward.
37. The graph of $y = |x|$ is reflected in the x-axis, stretched by a factor of 3, then translated 7 units to the right and 9 units upward.
38. The graph of $y = x$ is stretched by a factor of 2, reflected in the x-axis, then translated 8 units downward and 6 units to the left.
39. The graph of $y = \sqrt{x}$ is stretched by a factor of 3, translated 5 units upward, then reflected in the x-axis.
40. The graph of $y = x^2$ is translated 13 units to the right and 6 units downward, then reflected in the x-axis.

Use transformations to graph each function.

41. $y = \sqrt{x - 1} + 2$
42. $y = \sqrt{x + 5} - 4$
43. $y = |x - 1| + 3$
44. $y = |x + 3| - 4$

45. $y = 3x - 40$

46. $y = -4x + 200$

47. $y = \frac{1}{2}x - 20$

48. $y = -\frac{1}{2}x + 40$

49. $y = -\frac{1}{2}|x| + 40$

50. $y = 3|x| - 200$

51. $y = -\frac{1}{2}|x + 4|$

52. $y = 3|x - 2|$

53. $y = -(x - 3)^2 + 1$

54. $y = -(x + 2)^2 - 4$

55. $y = -2(x + 3)^2 - 4$

56. $y = 3(x + 1)^2 - 5$

57. $y = -2\sqrt{x + 3} + 2$

58. $y = -\frac{1}{2}\sqrt{x + 2} + 4$

Determine whether the graph of each function is symmetric about the y-axis or the origin. Indicate whether the function is even, odd, or neither.

59. $f(x) = x^4$

60. $f(x) = x^4 - 2x^2$

61. $f(x) = x^4 - x^3$

62. $f(x) = x^3 - x$

63. $f(x) = (x + 3)^2$

64. $f(x) = (x - 1)^2$

65. $f(x) = \sqrt{x}$

66. $f(x) = |x| - 9$

67. $f(x) = x$

68. $f(x) = -x$

69. $f(x) = 3x + 2$

70. $f(x) = x - 3$

71. $f(x) = x^3 - 5x + 1$

72. $f(x) = x^6 - x^4 + x^2$

73. $f(x) = |x - 2|$

74. $f(x) = (x^2 - 2)^3$

75. $f(x) = 1 + \frac{1}{x^2}$

76. $f(x) = \sqrt{9 - x^2}$

Match each function with its graph (a)–(h).

77. $y = 2 + \sqrt{x}$

78. $y = \sqrt{2 + x}$

79. $y = \sqrt{x^2}$

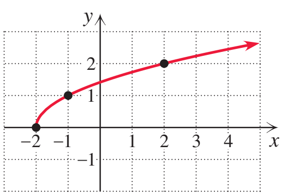
80. $y = \sqrt{\frac{x}{2}}$

81. $y = \frac{1}{2}\sqrt{x}$

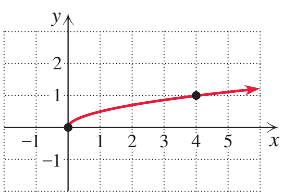
82. $y = 2 - \sqrt{x - 2}$

83. $y = -2\sqrt{x}$

84. $y = -\sqrt{-x}$



(a)



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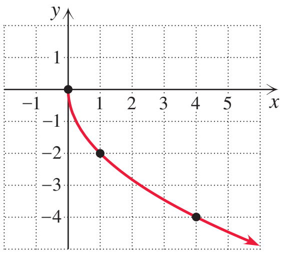

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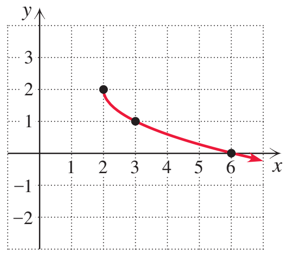

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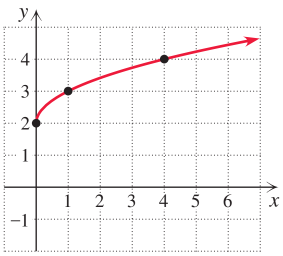

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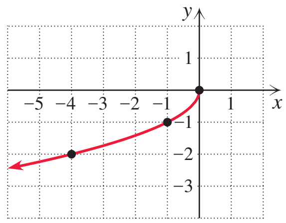

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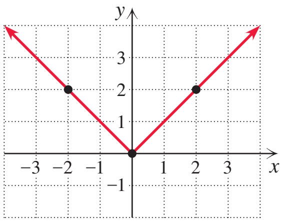
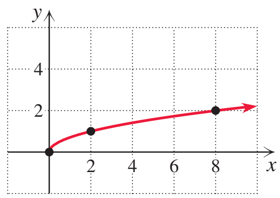


(g)



(h)

MODELING

Solve each problem.

85. *Across-the-Board Raise* Each teacher at C. F. Gauss Elementary School is given an across-the-board raise of \$2000. Write a function that *transforms* each old salary x into a new salary $N(x)$.

86. *Cost-of-Living Raise* Each registered nurse at Blue Hills Memorial Hospital is first given a 5% cost-of-living raise and then a \$3000 merit raise. Write a function that *transforms* each old salary x into a new salary $N(x)$. Does it make any difference in which order these raises are given? Explain.

87. *Unemployment Versus Inflation* The Phillips curve shows the relationship between the unemployment rate x and the inflation rate y. If the equation of the curve is $y = 1 - \sqrt{x}$ for a certain third world country, then for what value of x is the inflation rate 50%?

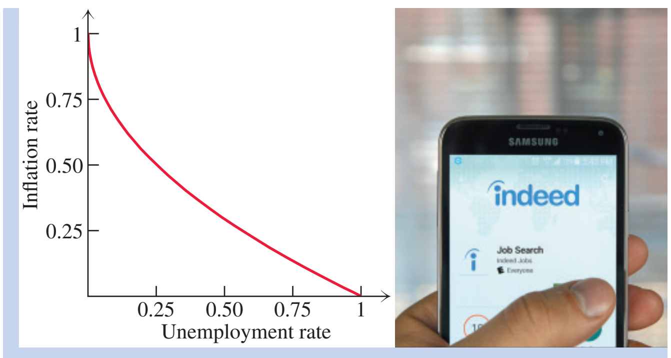


Figure for Exercise 87

88. *Production Function* The production function shows the relationship between inputs and outputs. A manufacturer of custom windows produces y windows per week using x hours of labor per week, where $y = 1.75\sqrt{x}$. How many hours of labor are required to keep production at 28 windows per week?

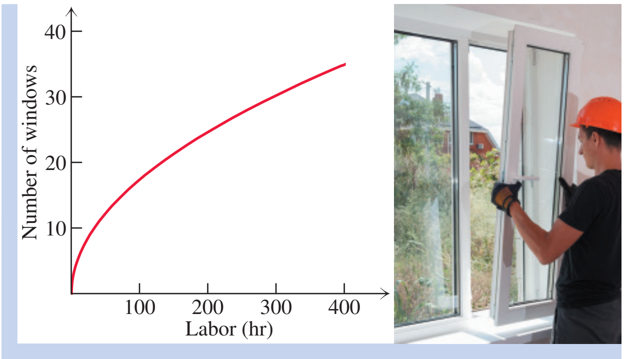


Figure for Exercise 88

WRITING/DISCUSSION

89. Graph each pair of functions (without simplifying the second function) on the same screen of a graphing calculator and explain what each exercise illustrates.

a. $y = x^4 - x^2,\ y = (-x)^4 - (-x)^2$

b. $y = x^3 - x,\ y = (-x)^3 - (-x)$

c. $y = x^4 - x^2,\ y = (x + 1)^4 - (x + 1)^2$

d. $y = x^3 - x,\ y = (x - 2)^3 - (x - 2) + 3$

90. Graph $y = x^3 + 6x^2 + 12x + 8$ on a graphing calculator. The graph is a transformation of the graph of a simpler function. What is the transformation?

REVIEW

91. Find the equation of a circle with center (0, 0) and radius 1.

92. Find the equation of a horizontal line through (0, 5).

93. Find the equation of a vertical line through (4, 0).

94. What equation expresses the area of a square as a function of its side?

95. Find $f(6)$ and $f(-x)$ if $f(x) = 2x^2 - 3x$.

96. Find a if $f(a) = 9$ and $f(x) = 2x^2 + 1$.

OUTSIDE THE BOX

97. *Door Prizes* One thousand tickets numbered 1 through 1000 are placed in a bowl. The tickets are drawn for door prizes one at a time without replacement. How many tickets must be drawn from the bowl to be certain that one of the numbers on a selected ticket is twice as large as another selected number?

98. *Completing the Rectangle* A rectangle has opposite vertices at $(1, -1)$ and $(3, 5)$. The other two vertices lie on the line $y = 2$. Find their coordinates.

P.3 POP QUIZ

1. What is the equation of the curve $y = \sqrt{x}$ after it is translated 8 units upward?

2. What is the equation of the curve $y = x^2$ after it is translated 9 units to the right?

3. What is the equation of the curve $y = x^3$ after it is reflected in the x-axis?

4. Find the domain and range for $y = -2\sqrt{x - 1} + 5$.

5. If the curve $y = x^2$ is translated 6 units to the right, stretched by a factor of 3, reflected in the x-axis, and translated 4 units upward, then what is the equation of the curve in its final position?

6. Is $y = \sqrt{4 - x^2}$ even, odd, or neither?

P.4 Compositions and Inverses

We are composing two functions when the output of one function is used as the input for a second function. If the second function undoes what the first function did, the functions are inverse functions. In this section we study compositions and inverses of the functions of algebra. We will see compositions and inverses throughout trigonometry as well.

Composition of Functions

If w is a function of t and z is a function of w, then z is a function of t. The function in which z is determined from t is the *composition* of the other two functions.

EXAMPLE 1 **Composition of functions given as formulas**

Express the area of a circle as a function of its diameter.

Solution

The equation $r = d/2$ expresses the radius as a function of the diameter. The equation $A = \pi r^2$ expresses the area as a function of the radius. By substituting $d/2$ for r we get

$$A = \pi \frac{d^2}{4},$$

which expresses the area as a function of the diameter.

TRY THIS. The diameter of a circle is a function of the radius $(d = 2r)$, and the radius is a function of the circumference $(r = C/(2\pi))$. Express d as a function of C.

In Example 1 it was merely a matter of substitution to find the composition of two functions that are given as formulas. The composition of two functions given in function notation is a function that is defined as follows.

Definition: Composition of Functions

If f and g are two functions, the **composition** of f and g, written $f \circ g$, is a function that is defined by the equation

$$(f \circ g)(x) = f(g(x)),$$

provided that $g(x)$ is in the domain of f. The composition of g and f is written $g \circ f$.

Note that to find $(f \circ g)(x)$ using function notation, we substitute $g(x)$ for x in the function f.

EXAMPLE 2 **Evaluating compositions defined by function notation**

Let $f(x) = \sqrt{x}$, $g(x) = 2x - 1$, and $h(x) = x^2$. Evaluate each expression.

a. $(f \circ g)(5)$ **b.** $(g \circ h)(x)$ **c.** $(h \circ g)(x)$

Solution

a. The expression $(f \circ g)(5)$ means to apply f to $g(5)$:

$$\begin{aligned} (f \circ g)(5) &= f(g(5)) && \text{Definition of composition} \\ &= f(9) && \text{Since } g(5) = 2 \cdot 5 - 1 = 9 \\ &= \sqrt{9} \\ &= 3 \end{aligned}$$

b. The expression $(g \circ h)(x)$ means to apply g to $h(x)$:

$$\begin{aligned} (g \circ h)(x) &= g(h(x)) && \text{Definition of composition} \\ &= g(x^2) && \text{Since } h(x) = x^2 \\ &= 2x^2 - 1 && \text{Substitute } x^2 \text{ for } x \text{ in } g(x) = 2x - 1. \end{aligned}$$