William B. Brower, Jr.

A PRIMER IN FLUID MECHANICS

Dynamics of Flows in One Space Dimension

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Dedication

to

The memory of my teacher, colleague, and friend

Joseph V. Foa

Acknowledgment

In the preparation of the seemingly endless number of drafts of this text I wish to acknowledge the meticulous work of Marjorie Baldwin, Mary Ellen Frank, Marilyn De Maria, and Victoria Lee.

The figures were computerized by the skills of Jamison Pond.

I had frequent occasion to call on the Reference Staff of the RPI Folsom Library, and they never failed to produce a requested book, technical paper, or other reference, for which I express my sincere gratitude.

A generation or more of RPI aeronautical engineering students suffered through the early drafts. I received from them a number of useful criticisms, which have resulted in improvements, and I was also the beneficiary of notification by them of a variety of existing errors, the bane of most authors.

The late Professor Joseph V. Foa read the early version of Chapters 3 through 5 and provided much informed criticism, all of which has been incorporated in the final version.

I wish also to thank my wife Yolanda who provided long-term encouragement in this endeavor and furnished much-needed support in getting out the final manuscript.

Epigraph

Although in classical and medieval times there was a certain amount of interest in hydrodynamics and more especially in hydrostatics, as is testified by the law of hydrostatics, and although this knowledge was advanced by the work of Stevin, Galileo and Newton, it is Leonhard Euler who is justly recognized as the father of hydrodynamics. To him we owe our clear ideas on fluid pressure, and it was through his grasp of this fundamental concept that he later propounded the equations of motion bearing his name. He was a great theoretical mathematician, yet he brought his genius to bear on such technical matters as the construction of turbines.

But the great growth in technical achievement which began in the nineteenth century left scientific knowledge far behind. The multitudinous problems of practice could not be answered by the hydrodynamics of Euler; they could not even be discussed. This was chiefly because, starting from Euler's equations of motion, the science had become more and more a purely academic analysis of the hypothetical frictionless "ideal fluid." This theoretical development is associated with the names of Helmholtz, Kelvin, Lamb and Rayleigh.

The analytical results obtained by means of this so-called "classical hydrodynamics" usually do not agree at all with the practical phenomena. To such extremely important questions as the magnitude of the pressure drop in pipes, or the resistance of a body moving through the field, theoretical hydrodynamics can only answer that both pressure drop and resistance are zero! Hydrodynamics thus has little significance for the engineer because of the great mathematical knowledge required for an understanding of it and the negligible possibility of applying the results. Therefore the engineers—such as Bernoulli, Hagen, Weissbach, Darcy, Bazin and Boussinesq—put their trust in a mass of empirical data collectively known as the "science of hydraulics," a branch of knowledge which grew more and more unlike hydrodynamics....

...Theoretical hydrodynamics seemed to lose all contact with reality; simplifying assumptions were made which were not permissible even as approximations. Hydraulics disintegrated into a collection of unrelated problems; each individual problem was solved by assuming a formula containing some undetermined coefficients and then determining these to fit the facts as well as possible. Hydraulics seemed to become more and more a science of coefficients.

Toward the end of the nineteenth century and the beginning of the twentieth century a new critical spirit appeared in both sciences...there grew up, among the [students of fluid mechanics] a more realistic attitude...having for its object a restoration of the unity between pure and applied science....

[Here] the relation between theory and practice will always be kept in the foreground. Theory will not be detached from the facts of experience but will be put in its proper

relation to them. Experimental results, on the other hand, will be considered in their relation to the fundamental laws and underlying theory.

From the Introduction to *Fundamentals of Hydro- & Aeromechanics*, Vol. 1, by L. Prandtl and O. G. Tietjens, 1934.

Preface

When I first began taking graduate courses in fluid mechanics, in the 1950s, I observed that the books I used as an undergraduate student were never referred to by the instructors and, in fact, were almost useless since they provided no bridge to the more-advanced topics.

This lack highlighted a wider gap. The undergraduate program existed in a historical vacuum. Concepts, theories, crucial experiments were often presented as though they were stand-alone entities, created "yesterday." Mention of the creators of these scientific and engineering disciplines was at most incidental, and provided little or no insight into how they developed.

Therefore, some years ago, while commencing work on the manuscript for this text, I determined to provide a moderate historical context that would be appreciated by the more curious student. Thus, I have identified many references—some of them the great engineering or science classics—to which the individual can immediately turn for further background, if desired. A second, particular goal is to provide to the working engineer a quick refresher or a ready introduction to the subjects within, and a source for references to standard works at an advanced level.

Although this is an introductory text devoted to flows in one space dimension, I have attempted, where possible, to tie these simpler flow models to their more complex counterparts in two or three dimensions, so that the interested individual can seek them out in the advanced literature. For example, in Chapter 1, on fluid properties, the molecular nature of fluids and its relation to the macroscopic view is dealt with on the lowest and most elemental level, but the student is also referred to the classic, and eminently readable, works of Sir James Jeans, and other more modern works, for proper treatments of the subject.

Inevitably, in an introductory book, most of the major areas covered are conventional. However, I have not hesitated to include advanced applications where warranted. In Chapter 2, on fluid statics, the analysis of the stability of a floating parabolic section is new, and provides an analytical challenge even for the superior student. In the same chapter I have included a treatment of the *U.S. Standard Atmosphere* because of its interest in applications related to the space program. Chapter 2 concludes with a short discussion of surface tension.

Some introductory books on fluid mechanics ambitiously start with the derivation of the basic equations in full tensor notation. (G. I. Taylor, one of the great names in fluid mechanics of the 20th century, once referred to such authors as the "i,j,k-guys," as contrasted to the "x,y,z-guys.") However, perusal of the applications presented in such texts reveals that most could have been just as effectively treated, and much more simply, by the one-dimensional approach followed herein. This should not be taken as a criticism of the importance of tensors, however.

Chapter 3 introduces the equations of fluid dynamics in some generality, although in one space dimension. The substantial derivative is introduced early as one of the time derivatives most frequently encountered in fluid mechanics, with emphasis on its physical significance. A number of examples are provided. The one-dimensional approach lends itself to the combining of the first law of thermodynamics with the dynamic equation to produce the energy equation, in a straightforward way. The equations for the specialized cases of constant-density flow and compressible flow of a perfect gas follow readily, and without the suggestion that they are really two different subjects. Special attention is given to the question of viscous dissipation in a flow.

Chapter 4 deals with conventional applications in constant-density flow including several that employ the integrated momentum equation, one of which involves the more-complicated nonsteady form. Several deal with the concept of efficiency of fluid machinery.

In the fifth chapter, after a discussion of units, dimensions, and standards, I explore the subject of dimensional analysis starting with a real-life example from the work of Theodore von Kármán. Several examples are then used to show how some of the more commonly encountered π-ratios appear routinely in a dimensional analysis. The associated topic of inspectional analysis is then introduced to show that when a differential equation governing a physical phenomenon is available, an inspectional analysis is capable of demonstrating dependency upon the appropriate π-ratio "naturally" and without necessarily integrating the equation, an important advance. Examples are given that illustrate the virtues of correlating data in a nondimensional format and further demonstrate certain limitations on model testing.

In Chapter 6, a brief history is given of the early work on viscous flow in pipes, emphasizing the fact that over a period longer than two millennia hydraulics engineers had achieved considerable practical success in building water-distribution systems, with access to only elemental, or empirical, methods of quantitative flow prediction. Starting with the formulation of the Darcy–Weisbach law for head loss, we follow the development of the subject: the Navier–Stokes equations; the experiments of Hagen and Poiseuille on laminar flow and its theoretical confirmation by Stokes; the demonstration of transition from laminar to turbulent flow, and the criterion therefore, by Reynolds; the use of dimensional analysis to correlate pipe flow date by Blasius; Prandtl's law of turbulent flow in smooth pipes; the remarkable experiments of Nikuradse on artificial pipe roughness and velocity profiles in turbulent flow; and finally the correlation of data on commercial pipes by Moody. Two sections deal with the problem of determining pumping power in hydraulic systems. The treatment of viscous flow in pipes and fittings concludes with a prescription reducing the analysis of flow in hydraulic loops to a routine.

In 1904 Prandtl formulated the concept of the boundary layer flow that enabled researchers to deal with a flow in which the viscous effects are restricted to a small region near the body surface. The current text deals with boundary layers only in a qualitative way.

For the curious student I have included an analysis of a flow in which the duct contour interacts with the flow—the case of a pressurized flow in an elastic tube. This problem introduces the concept of an approximate solution by expanding a

function in a series involving a small parameter, a technique that is further exploited in Chapter 8.

Chapter 7 starts with a review of elementary thermodynamic relations applied to gases. The first major flow discussed celebrates once again the keen insight of Reynolds who resolved the perceived paradox of a gas flow from a reservoir through an orifice which attained a maximum flow rate independent of the value of the pressure applied on the downstream side of the orifice, as long as the pressure was lower than some fraction of the reservoir pressure. His analysis shows that at a station where the flow speed equals or exceeds the local speed of sound the downstream regime can no longer "communicate" imposed pressure changes to the upstream flow regime.

This leads naturally to the study of the de Laval convergent–divergent nozzle and, eventually, to the appearance of shock waves—which are treated as finite discontinuities in the flow—in the divergent section of the nozzle. All the applicable relations are appreciably simplified by the introduction of the Mach number as a flow parameter. For the more perceptive student who might wonder about the consistency of discontinuities appearing in a supposedly continuous substance, I have included the analysis of Becker for the calculation of the (very thin!) thickness of a shock when treated by continuum theory. This is followed by discussions of Fanno flow and isothermal flow.

In the final chapter, Chapter 8, the behavior of nonsteady flows is explored. The first group of applications deals with the draining of "inviscid" liquids from ducts of fixed geometry. Certain of these problems go back to Newton himself (who got it wrong) and to other pioneers in fluid mechanics, especially Bernoulli (who got it right). Several of these solutions are presented here for the first time, and comparisons with experiment are made in several cases.

In most of these examples the analysis involves a second-order, ordinary differential equation, usually nonlinear, which must be integrated. In the case of draining a slender, conical reservoir, the equation is highly nonlinear. Its solution is approximate, and requires the use of the powerful technique involving a singular perturbation analysis.

The remainder of the applications deals with compressible flows in uniform ducts. Since the governing equations are a pair of coupled nonlinear partial differential equations, it is necessary to introduce the theory of hyperbolic functions for functions of two variables. With the advent of the computer age and the use of packaged software programs, it is vital for the analyst to be able to distinguish among hyperbolic, parabolic, and elliptical differential equations. The criterion depends on whether or not the characteristics—lines along which disturbances are transmitted—are "real."

The techniques involved are spelled out in detail, with elementary applications. These include the analysis of an isentropic expansion pulse generated by the impulsive motion of a piston and the propagation of an isentropic finite-amplitude pulse that leads to the inception of a shock wave and a calculation for the length of the duct required for the shock to form.

The chapter concludes with a short section on acoustical theory. The acoustic equations are deduced from the preceding characteristic equations of gas dynamics,

by imposing approximations that require that flow disturbances satisfy certain "smallness" criteria. The resulting governing relation turns out to be the wave equation of classical physics, which is linear. One elementary application is given, and some of the technical standards used in evaluating acoustical phenomena are described.

An extensive group of problems for the first seven chapters is provided. There are few routine exercises. Most of the problems are intended to provoke thought from students and to require ulitization of a variety of analytical skills in the process of solution. Some of the problems stem from real-life situations, and, where appropriate, a modest number of them have been annotated.

To teachers who may contemplate adoption of this text, I note that for a number of years I utilized xeroxed copies of the preliminary manuscript (lacking Chapter 8) for a one-semester, 3-credit course, at Rensselaer Polytechnic Institute. The course outline included Chapters 1 through 6, omitting certain sections, e.g., Sections 2.14 and 6.14, which varied somewhat as the manuscript grew and the needs changed.

W.B. Brower, Jr.
Troy, New York
April, 1998

Table of Contents

Chapter 2

1 Fluid Properties — Kinetic Theory of Gases

At last you must admit, confess
That tiny things there are of nature indivisible.
Since this is so, this, too you must allow
That these atomic shapes are made of solid stuff,
and will for evermore endure.

Lucretius (98–55 B.C.E.)
Set in English verse by
Allen Dewes Winspear (1955)

If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generation of creatures, what statement would contain the most information in the fewest words? I believe it is the atomic hypothesis...that all things are made of atoms, little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another.

Richard P. Feynman et al. (1963)

1.1 INTRODUCTION

In order to discuss fluid mechanics it is desirable first to indicate in an unambiguous way what is meant by a fluid. Intuitively, we all recognize the difference between gases and liquids, which are both fluids and which are two of the three commonly identified states of matter,* solids being the third.

Fluid Deformation

The crucial distinction between a solid and a fluid is that a fluid will deform continuously when subject to a shear force, however small. In Figure 1.1-1 a quantity of matter, originally rectangular in cross section, is subject to a shear force such that the lower boundary remains fixed. Equilibrium, of course, requires that an equal and opposite shear force is applied on the lower boundary. The deformation is measured by the angle ϑ between the left boundary and the vertical.

In an elastic solid (for example, a rectangular sponge to the top and bottom of which are glued thin plates) ϑ would achieve a fixed angle and not vary thereafter as long as the force system remains in place. However, a fluid would continue to

* A *plasma*, consisting of a collection of charged particles in the gaseous state, is sometimes said to be the fourth state of matter.

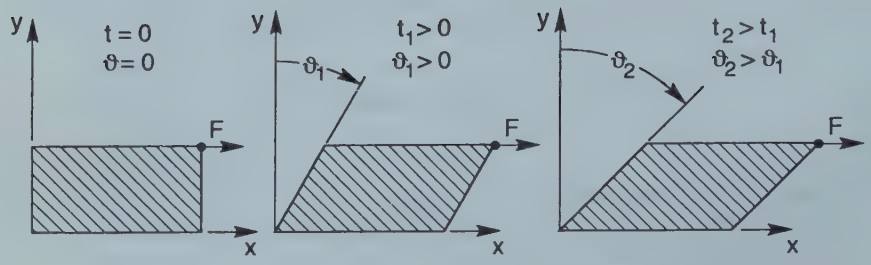

FIGURE 1.1-1 Example of continuous deformation of a fluid element under a shearing force.

deform as long as the forces are imposed. The time derivative of the angle $d\vartheta/dt \equiv \dot{\vartheta}$ is the *rate of shear deformation.*

Nonsimple Fluids

But there are fluids whose behavior is complex and for which the preceding elementary criterion becomes inadequate. For example. there are *thixotropic* materials such as some paints and jellies, which, having stood for some time, behave as elastic bodies. That is, as long as the forces are not too great, the bodies will return to their original shape when the shearing force is removed. However, if the material is stirred up, it behaves like a fluid.

On the other hand, there are other substances, ordinarily considered as solids, that may also exhibit fluid behavior. For example, at ordinary temperatures, pitch will fracture if struck sufficiently hard. Yet, if it is subjected to shearing action, it tends to deform, albeit very slowly. The rate of deformation is so small it may take weeks or months to detect the deformation by eye. Similarly, glass deforms under shearing action, although it may take centuries to achieve a visually observable deformation.

Modern chemistry has produced a number of remarkable substances — *polymers*, which may display simultaneously certain characteristics of solids and fluids. Nor surprisingly, the formulation of a mathematical model that can account for their behavior becomes more difficult than for more elementary substances.

Simple Fluids

This text will concentrate on *simple fluids*, which are defined as materials that, when subjected to a shear force, however small, respond by exhibiting a finite rate of deformation. For engineering purposes this covers an extremely wide range of applications. For example, at moderate temperatures and pressures, air and water can be treated as simple fluids. Even for fluids not covered by this definition, knowledge of the behavior of a simple fluid is useful as a point of departure for studying the behavior of more complex fluids.

To understand why fluids differ from solids, and how gases differ from liquids, it is necessary to recognize that matter is molecular in nature. The rest of this chapter gives a qualitative description of how fluids differ from solids from the molecular viewpoint, followed by a cursory treatment of how *macroscopic* properties of a gas,

such as pressure, temperature, and viscosity, can be explained from a molecular viewpoint.

1.2 THE MACROSCOPIC VIEW OF MATTER

Although the ancients recognized that matter consisted of extremely small particles, this was not conclusively established until the 19th century by various investigators employing a wide range of experimental techniques.

The Solid State

In the solid state, for example, the atoms of a crystal, are densely packed, often with their centers at the intersections of a regular lattice. It is known that the atoms are held in position by the combined effects of the force fields of the surrounding atoms. Yet, the individual atoms are not in fact at rest. Alder and Wainwright (1959) illustrated this fact by a calculated plot — a portion of which is shown in Figure 1.2-1 — in which the traces of the pathlines of 32 "hard" atoms (conceptually elastic spheres) in a square-grid lattice lying in a plane were plotted over a time interval involving about 3000 "collisions." The pathline for each atom repeatedly self-intersects at or near the mean position. For a sufficiently long time interval the plot of each path depicted would devolve into a smear, or blot, centered on the particle mean location in the crystal grid.

Recent developments in electron microscopy have since permitted this fact to be conclusively demonstrated. Figure 1.2-2a is a time-exposure photograph of an extremely thin slice of a zirconium oxide (ZrO_2) crystal, at a magnification of 300,000 times, taken on the Atomic Resolution Microscope by R. Gronsky at the Lawrence Berkeley Laboratory. The large, white, almost circular areas are the zirconium atoms. At the intermediate lattice points are the oxygen atoms, which are much less bright. The diffuseness of the image can be attributed partly to the resolution limit of the microscope, which is 0.18 nm, but, more importantly, to the motion of the molecules about their mean positions. Figure 1.2-2b indicates the scale of the lattice structure. For a description of the microscope see Gronsky (1984).

The Intermolecular Force Field

The intermolecular force field between any pair of atoms depends on the magnitude of the separation distance of the pair. The force is mutually attractive when the separation d exceeds some (extremely small) distance d_0 and the attractive force magnitude is approximately proportional to d^{-7}. When the separation distance is such that $d < d_0$ the force becomes extremely repulsive with magnitude approximately proportional to d^{-13}. Deductions from calculations compared with experiment indicate that the distance d_0 is the order of 3 to 4×10^{-10} m.

As the crystal is heated, the average speed of random motion of the ensemble of molecules increases. Thus, on the molecular level, the acquisition of heat results in an increase of the kinetic energy of random motion of the molecules. If the crystal continues to be heated, the average molecular speed continues to increase. However, at any instant, the speed of a given molecule may vary from zero to some speed

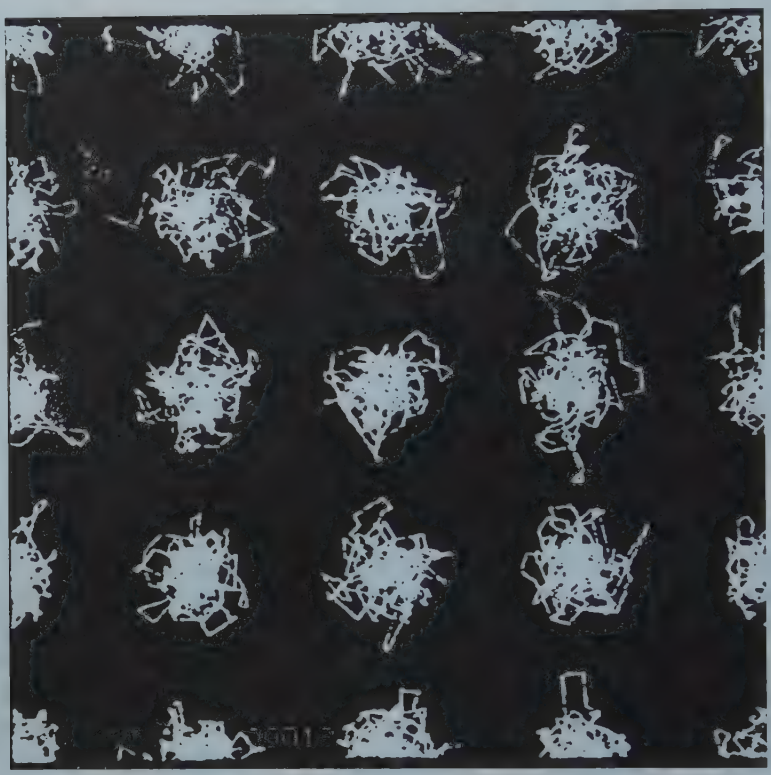

FIGURE 1.2-1 Plots of the theoretical pathlines of atoms in a crystal. (Calculated by Alder, B.J., and Wainwright, T., *J. Chem. Phys.*, 31(2), 459–466, 1959, and reprinted with permission.)

above which a molecule may have sufficient energy to move away from its position in the lattice. As the heating continues, this must occur and the crystal begins to melt.

1.3 THE LIQUID STATE

Melting

In the melting process heat is added, but the crystal temperature remains constant. For most substances the volume of the material increases, leading to a decrease in density. Water is exceptional in that the opposite occurs over a small temperature range above the melting temperature.

In the liquid state (the melt must now be confined by walls) there remain certain similarities to solids and certain crucial distinctions. In mechanical equilibrium (i.e., subject to no shearing forces) the molecules remain partially ordered. The packing spacing remains approximately the same as for the solid state. That is, a molecule may vibrate randomly about a mean position, but there is a degree of mobility that allows for possible interchange or rearrangement of individual molecules or even of large groups of molecules acting together. Thus, there is a continual reordering of

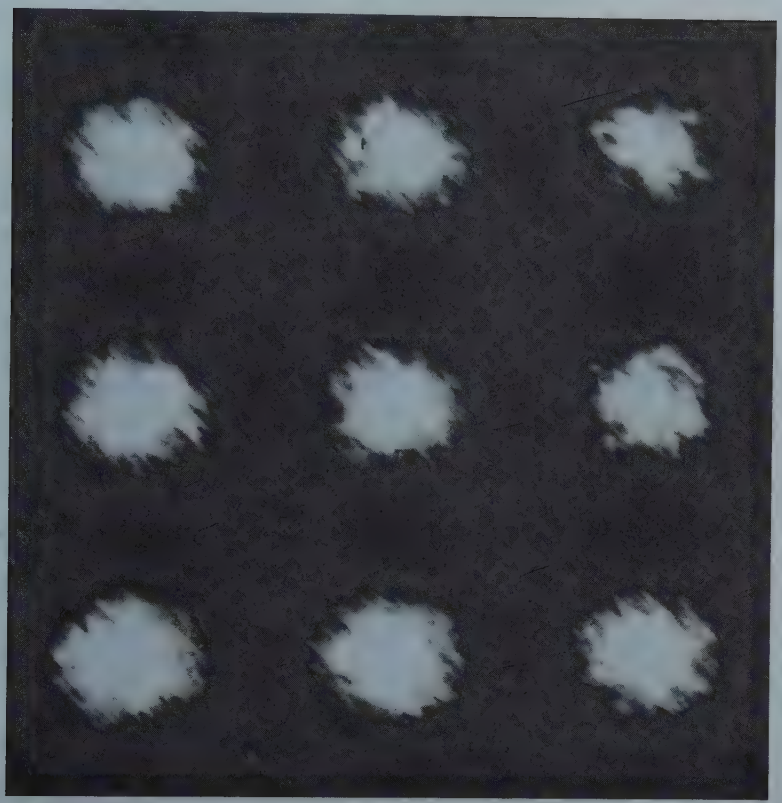

FIGURE 1.2-2a Electron microscope photograph of a slice of a crystal of zirconium oxide. (Photograph from Lawrence Berkley Laboratory, University of California courtesy of Ronald Gronsky. With permission.)

the structure. In both the liquid and solid state the average spacing of the molecules is the order of d_0.

Evaporation

If the heating process continues beyond the point at which the substance is fully liquefied, molecules near the free surface* may undergo a series of collisions** with their neighbors in which they acquire sufficient energy to escape from the force field exerted by the main body of a fluid. In such a case, these molecules may fly off into the space above the fluid. In the macroscopic sense this process is called *evaporation.* If the liquid is in an enclosed chamber, the liquid phase establishes, for any specified temperature, an equilibrium with its gas phase above the free surface, in which the rate at which molecules are emitted by the free surface in evaporation just equals

* The free surface is the boundary between the gas and the liquid.

** The reader is cautioned that no pretense is made herein to deal with the structure of a molecule on the level of quantum mechanics. For the purpose of this book it is adequate to think of a molecule as an elastic sphere of extremely small diameter.

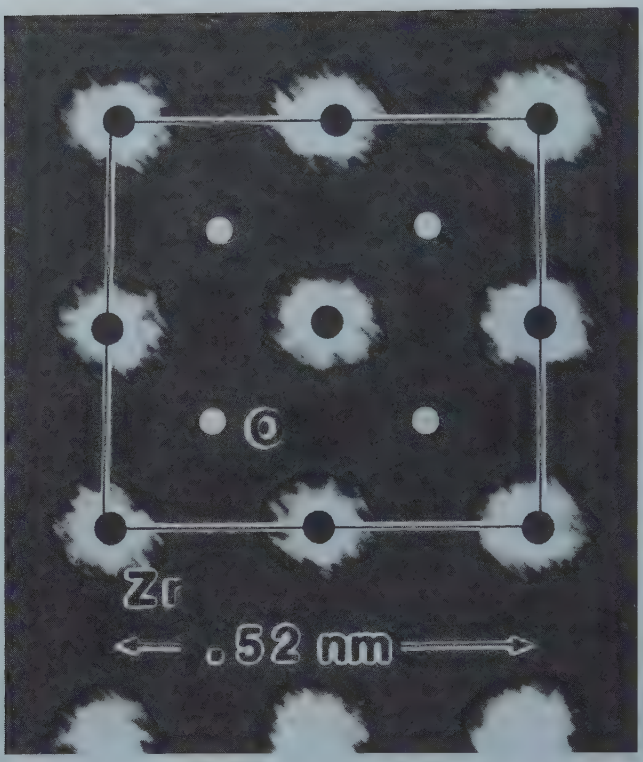

FIGURE 1.2-2b The scale of a ZrO_2 crystal. (Photograph from Lawrence Berkley Laboratory, University of California courtesy of Ronald Gronsky. With permission.)

the rate at which molecules are received from the gas phase in *condensation*. As is well known, once the equilibrium temperature is established, the equilibrium pressure is also automatically fixed at a unique value.

Viscosity

From the macroscopic view, a liquid, when subject to shearing action, tends to deform as long as the shearing force remains imposed. For a fixed shear force the rate of deformation depends on the liquid temperature; that is, as the temperature increases, the rate of deformation increases, which means that the viscosity decreases.* The state of the kinetic theory of liquids is not sufficiently developed, however, to allow complete prediction of the fluid properties from the molecular structure of the liquid. This question will be revisited in a later section.

As we continue to add heat to the liquid, its temperature and pressure rise, its density and viscosity decrease, and the fraction of the substance evaporated increases. Eventually, all the material is evaporated into the vapor phase and we are then dealing solely with a gas.

* A quantitative expression for viscosity is developed in Section 1.8.

1.4 ON THE KINETIC THEORY OF GASES

The kinetic theory of gases was extensively developed in the 19th century and the first half of the 20th by a host of investigators, which includes some of the greatest names in science. For a proper treatment of the subject the reader is referred to Jeans (1952), Loeb (1934), and Hirschfelder et al. (1954), in ascending order of comprehensiveness. A comprehensive, modern review of gas kinetic theory can be found in Touloukian et al. (1975). For the nonspecialist, Hildebrand (1963) provides an excellent introduction. A more recent book by Tabor (1991) covers a broader range of the properties of matter, also on a level conducive to the nonspecialist. In what follows I take the liberty to make occasional gross oversimplifications since space does not permit rigorous treatment. For the average reader this will not present a serious problem if he or she keeps in mind that the primary objective is to show that many of the fundamental concepts of fluid mechanics (of gases) can be considered to have a mechanical origin.

1.5 MASS DENSITY

Definition of Density

Consider a control volume which is a cube of side ℓ with permeable walls. At a fixed instant of time inside the volume, let there be N molecules each of mass m. Then the average *number density* is defined* as

$$\nu \equiv N/\ell^3. \tag{1.5-1}$$

The average *mass density* is denoted ρ and is given by

$$\rho \equiv Nm/\ell^3 = \nu\bar{m}u, \tag{1.5-2}$$

where u is the *unified atomic mass unit.*** For future reference we note that the quantity *specific volume*, denoted v, is the inverse of the density, i.e.,

$$v \equiv 1/\rho. \tag{1.5-3}$$

Concept of a Continuum

In a fluid mechanical system we shall find it desirable to introduce the concept of a *continuum*, i.e., the idea that in treating ordinary flow problems we can ignore the fact that a fluid consists of finite particles in which the masses are concentrated, separated by distances that are large relative to molecular diameter. In fact, we shall

* The symbol $\equiv$ denotes a definition as distinguished from an equals sign, which refers to an equation.
** In the International System of Units (SI) the unified atomic mass is fixed at $1u = 1.660\ 53 \times 10^{-27}$ kg. The mass of a molecule, then, is given by $m = (1.660\ 53 \times 10^{-27})\bar{m}$ kg, where $\bar{m}$ is the *molecular weight* (actually a pure number).

postulate, under conditions to be stated, that density is an analytic point function of space and time, i.e., $\rho = \rho(\mathbf{r}, t)$, where $\mathbf{r}$ is the position vector. But the continuum density cannot be obtained from Equation 1.5-2 by letting the control volume — centered at an arbitrary, fixed point — shrink to zero; i.e., we cannot put

$$\rho(\boldsymbol{r}, t) \equiv \lim_{\ell \to 0} Nm/\ell^3 .$$

To see this, imagine an enormous control volume of 10 km on the side, with centroid at 5 km. According to the *U.S. Standard Atmosphere, 1976* (Dublin, et al. 1976; see Table 2.16-2), the number densities at three different heights are

Altitude (km)	ν (m^{-3})
Sea level	2.547×10^{25}
5	1.531×10^{25}
10	0.860×10^{25}

The nonlinearity of the number distribution requires that the integrated average of ν over 10 km must exceed that at the median altitude of 5 km. As we shrink the cube side, keeping the centroid at 5 km, the plot of the number density (and equally true, the density) must appear something like that shown in Figure 1.5-1. As ℓ decreases to some value denoted ℓ', the volume becomes sufficiently small that the instantaneous calculation of the density starts to display a time-dependent effect due to the randomness of the number of molecules within the cube. The smaller the volume, the larger the possible deviations from the mean value become. Eventually, the macroscopic concept of density breaks down and we must resort to an alternative limiting condition.

To get a preliminary, though crude, idea of the conditions under which it is reasonable to express the density as an analytic function — which is to say that we are dealing with a continuum — we suppose that within a cube of side ℓ' there must be a minimum of N molecules, where N is to be determined by some as yet unspecified criterion. For a fluid of number density ν the side of the cube would be $\ell' = (N/\nu)^{1/3}$.

If $N = 10^6$ molecules, the corresponding value of ℓ' in air at sea level is about 3.4×10^{-7} m. At an altitude of 10^3 km, according to the *U.S. Standard Atmosphere, 1976*, where $\nu = 5.422 \times 10^4$ m^{-3}, then $\ell' = 1.2 \times 10^{-2}$ m = 1.2 cm. By contrast, in liquids, the cube edge necessary to contain 10^6 molecules is only about $\ell' = 3.5 \times 10^{-8}$ m.

Suppose that we want to install a pressure tap on some aerodynamic surface. We have the instinctual feeling if there is a sufficient number of molecules present, the tap should yield valid readings, from the continuum viewpoint. Perhaps we should compare the tap diameter to ℓ'. The diameter of a really fine pressure tap or sensing element could be the order of 0.012 5 cm = 1.25×10^{-4} m. Continuing our example, we see that the pressure tap diameter is much larger than ℓ' for the sea-level measurement and much smaller at 10^3 km.

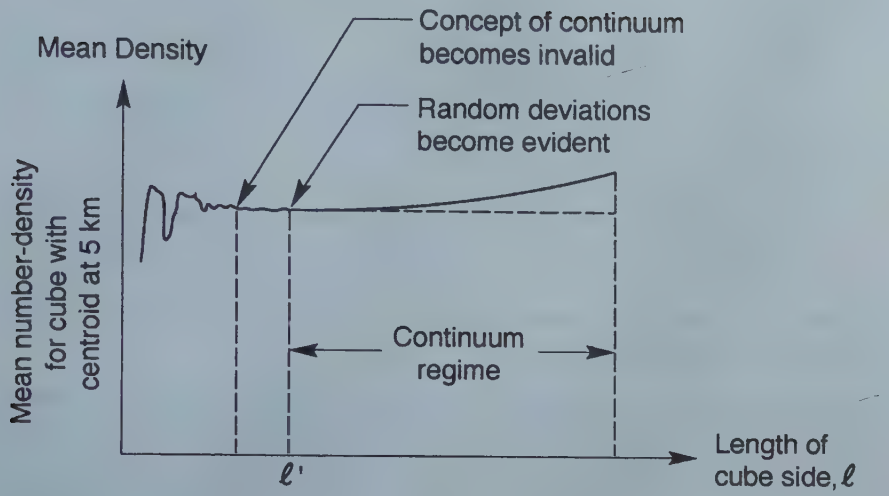

FIGURE 1.5-1 Qualitative explanation of breakdown of density function as analytic function of space and time.

However, the procedure involving 10^6 molecules is completely arbitrary. There is no way to formulate a valid criterion depending on a specific number of molecules. In Section 1.7, therefore, a different approach is used, which has a sound scientific foundation and which allows us to quantify the continuum criterion. The principal virtue, of course, is that it allows us to consider the density function — all state functions, in fact — to be treated as analytic. The sole exception occurs for shock waves, which are treated as discontinuities across which the density, pressure, velocity, etc. may involve finite jumps. Even so, as shown in Section 7.14, the concept of a shock as an infinitely thin discontinuity is a convenient analytical fiction. Of course, the idea of a continuum is itself a fiction, and aerodynamicists must keep in mind that continuum theories must not be applied to a physical body in violation of the criterion developed in the following pages.

1.6 PRESSURE IN A GAS

Consider a cubical container, Figure 1.6-1, with rigid walls of side ℓ in which there is a total of N molecules. Each molecule is considered to be an elastic sphere of mass m. These molecules are supposed to be moving randomly, and we denote the components of the average speeds* of the molecules as u, v, w, for the x-, y-, z-directions, respectively.

If there is true randomness, then, at any instant, $N/2$ of the molecules have an x-component of velocity to the right, the other half to the left. When a molecule (see Figure 1.6-1b) undergoes an elastic collision with a wall, we know from elementary mechanics that the tangential components of momentum remains

* The symbol u for the x-component of velocity should not be confused with the atomic mass unit, which is designated by the same symbol.

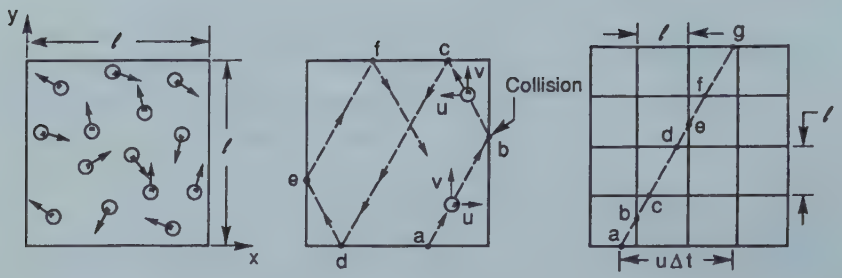

FIGURE 1.6-1 Pressure as a molecular momentum exchange: (a) single cell; (b) path with collisions; (c) honeycomb of cells.

unchanged, but that the component of momentum normal to the wall is reversed in sign. Following Jeans (1952), we can see that a simpler way to study the path of one molecule is to draw a honeycomb of identical cells of side ℓ. Then, the segments of the extended path on the right are exactly the same as the segments contained in the center sketch.

We can now make the following calculations:

x-distance traveled to right in time Δt $= u\Delta t,$

Number of collisions with wall perpendicular to x-direction, for one molecule $= u\Delta t/\ell,$

Total number of collisions with wall perpendicular to x-direction, for N molecules $= Nu\Delta t / \ell,$

Total number of collisions on wall facing left $= \frac{1}{2} Nu\Delta t / \ell,$

Total change of momentum of all molecules on wall facing left $= \left(\frac{1}{2} Nu\Delta t / \ell\right) 2mu,$

$= Nmu^2 \Delta t / \ell,$

$$\text{Force on left-facing wall} = \frac{\text{Tot. chge. momentum}}{\Delta t} = \frac{Nmu^2}{\ell}.$$

Thus, the pressure on the left-facing wall is defined as

$$p_x \equiv \frac{\text{Force}}{\text{Area}} = \frac{Nmu^2}{\ell^3},$$

which is the same as for the right-facing walls. By an identical procedure

$$p_y = \frac{Nmv^2}{\ell^3}, \quad p_z = \frac{Nmw^2}{\ell^3}.$$

The average pressure within the container is defined as the average of these three quantities, i.e.,

$$p \equiv \frac{1}{3}\left(p_x + p_y + p_z\right) = \frac{1}{3}\frac{Nm}{\ell^3}\left(u^2 + v^2 + w^2\right). \tag{1.6-1}$$

Definition of Mean-Free-Speed

The *mean-free-molecular speed* C is defined as

$$C^2 \equiv u^2 + v^2 + w^2. \tag{1.6-2}$$

In fact, in any realistic container the number of molecules is truly enormous (about 2.55×10^{25}) $m^{-3} \approx 7.21 \times 10^{23}$ ft^{-3} in air at standard atmospheric conditions). Further, there is actually a distribution of speeds for a gas in equilibrium ranging from a mean free speed of zero to infinity, according to a statistical law discovered by, and named for, the British physicist Maxwell. A rigorous analysis shows that C is the speed with the highest probability of occurrence.

Combining Equations 1.6-1 and 1.6-2, we have

$$p = \frac{2}{3}\frac{N\left(\frac{1}{2}mC^2\right)}{\ell^3}, \tag{1.6-3}$$

which demonstrates that pressure in a (monatomic) gas can be interpreted* as

$$p = \frac{2}{3}\frac{\text{Kinetic Energy of Random Motion}}{\text{Volume}}. \tag{1.6-4}$$

Furthermore, since kinetic energies are additive, the combined pressure of the mixture of two gases must equal the sum of the pressures exerted by the constituents of the mixture separately. This is *Dalton's law* of partial pressures. We further see that pressure varies inversely with the volume, which is *Boyle's law.* By introducing the definition of density, Equation 1.5-2,

$$p = \tfrac{1}{3}\rho C^2. \tag{1.6-5}$$

Equipartition of Energy

It can be shown rigorously, for a mixture of gases, that the average kinetic energy of a particle is the same for each constituent. Furthermore, since there is no reason to expect that there is any preferred direction in a container, the three average velocity components of the same gas must be equal, i.e.,

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* Equation 1.6-4 is strictly valid only for monatomic gases.

$$u^2 = v^2 = w^2 = C^2/3. \tag{1.6-6}$$

One third of the kinetic energy of random motion is associated with the mean velocity component for each of the three orthogonal space coordinates. This lack of preferred direction is called the principle of *equipartition of energy.*

Equation of State

If Equation 1.6-6 is compared with the *equation of state for a perfect gas,**

$$p = \rho \mathcal{R} T, \tag{1.6-7}$$

which results from the thermodynamic requirement (see Chapter 7) that a gas be thermally perfect, then

$$C^2 = 3\mathcal{R}T. \tag{1.6-8}$$

Clearly, then, we see that temperature is a measure of average kinetic energy of random motion of a particle.

1.7 THE MOLECULAR MEAN-FREE-PATH LENGTH

The Molecular Diameter

It was previously noted that in the process of evaporation a molecule flies off into the space above a liquid. In the gas phase the average spacing between gas molecules arranged in a cubical lattice is $v^{-1/3}$, the inverse cube root of the number density, a length which is generally very large compared with the molecular diameter. For molecules in motion the lattice spacing has little significance; a more meaningful quantity is the mean-free-path, which is described in the following paragraphs.

That a molecule has an effective diameter can easily be established for some substances. One way is to pour a known volume of a liquid onto the surface of another, immiscible liquid that is more dense. If the volume of the less-dense liquid is sufficiently small, it will spread out in a layer one molecular-diameter thick, which does not occupy the entire cross section of the container. This area is then measured. Its value, divided into the original volume, produces the desired molecular diameter.

More sophisticated techniques are required for gases. If it is assumed that the molecules are elastic spheres, then it is possible to deduce a theoretical value for the coefficient of viscosity and similarly, the coefficient of self-diffusion, both of which depend on the postulated molecular diameter. These macroscopic quantities

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* *The specific gas constant* $\mathcal{R}$ is related to the *universal gas constant* $\mathcal{R}^*$ such that $\mathcal{R} = \mathcal{R}^*/\overline{m}k$, where $\overline{m}$ is the molecular weight (actually a dimensionless number) and where k is the *molar mass*, of unit magnitude, but of units consistent with those of the universal gas constant, e.g., kg/kmol, gm/gm mol, slug/slug mol, etc. See Section 7.2 for further explanation and see Tables 2.16-2 and 7.2-1, respectively, for values of $\mathcal{R}$ and $\overline{m}$ for various gases.

TABLE 1.7-1
Molecular Diameters* Deduced from Viscosity and Self-Diffusion Calculations

Gas	Molecular diameter $\sigma \times 10^8$ (cm)	
	From Viscosity	From Diffusivity
Argon	3.64	3.47
Neon	2.58	2.42
Nitrogen	3.75	3.48

* This table should be read as an equation: $\sigma \times 10^8$ = Value from table, and similarly throughout.

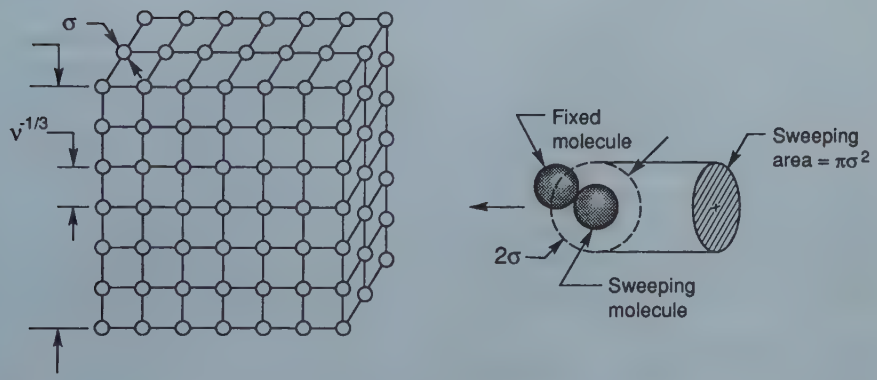

FIGURE 1.7-1 The calculation of mean-free-path length.

can be measured, allowing the effective diameter to be calculated. A few values taken from Hirschfelder et al. (1954), p. 545, are listed in Table 1.7-1.

Mean-Free-Path Length

In an ensemble of moving molecules, the *mean-free-path length* is defined as the average length a molecule travels between collisions with other molecules. If molecules are treated as elastic spheres the calculation is relatively simple. However, if molecules are treated as point masses, obeying the attractive/repulsive law of intermolecular forces previously mentioned, the word *encounter* better describes the interaction process, and the calculation gets quite involved.

Referring to Figure 1.7-1 we can get an approximate idea of the mean-free-path length as follows. Consider a volume of side ℓ containing N molecules of diameter σ. Suppose, for the current purpose, that the molecules are not in motion but are fixed at the lattice points of a cubical grid. If another molecule, also of diameter σ, moves through the volume to sweep it out without duplication, then we can consider a collision to occur any time the center of a fixed molecule falls within the sweeping

circle of diameter 2σ. Obviously, the total number of "collisions" is the same as the number of molecules in the lattice.

The sweeping area is $\pi\sigma^2$ and the total length required to sweep the volume is $\ell^3/\pi\sigma^2$. Thus, approximately, λ, the mean-free-path length between collisions, is given by

$$\lambda \equiv \frac{\ell^3/\pi\sigma^2}{N} = \frac{1}{\pi\nu\sigma^2}. \tag{1.7-1}$$

This result is surprisingly close to the result of rigorous analysis, which involves an infinity of molecules in random motion, and whose velocities vary according to Maxwell's law. The accurate calculation yields

$$\lambda = \frac{1}{\pi\sqrt{2}\nu\sigma^2}. \tag{1.7-2}$$

At standard atmospheric conditions $\lambda = 6.633 \times 10^{-8}$ m, whereas at a height of 10^6 m, $\lambda = 3.1 \times 10^6$ m. Jeans estimates that in interstellar space the mean free path is the order of 10^{11} km, and that in internebular space $\lambda = 10^{16}$ km, or about 1000 light-years.

It should now be clear that the characteristic length that represents the extent to which a gas is dilute or dense is λ the mean-free-path length, which depends on both the number density ν and the effective molecular diameter σ. It is λ and not the somewhat nebulous length ℓ' of Section 1.5, which we shall use in the criterion for the validity of continuum theory.

The Knudsen Number

The criterion is very simple. The mean-free-path length is compared with L, the characteristic dimension of a body immersed in a flow. If the body dimension is very large in comparison with λ, then the continuum concept (which implies that there is a distribution of molecular velocities in time/space governed by Maxwell's law and that collisions between molecules are important) is valid. If the mean-free-path length is equal to, or larger than the body dimension, then free-molecule theory, rather than continuum theory, must be employed. In free-molecule theory, collisions between molecules need not be considered. Density is computed (over a large volume) in the same way as before, but it is not considered to be an analytic function.

Thus, the mean-free-path and characteristic lengths are combined into the dimensionless parameter

$$\text{Kn} \equiv \frac{\text{Mean-free-path length}}{\text{Characteristic body dimension}} = \frac{\lambda}{L}, \tag{1.7-3}$$

named for M. Knudsen, who made a number of original and fundamental investigations of free-molecule flow, early in the 20th century; see Knudsen (1933).

The characteristic body dimension is usually apparent from the nature of the flow. If the flow is through a tube or hole, then L = diameter. For a space vehicle, then L = vehicle diameter, and so forth. The criterion used by the aerodynamicist divides the flow into regimes; i.e.,

$$\begin{aligned} &\text{Kn} < 0.1, \quad \text{Continuum regime;} \\ 0.1 \leq &\text{Kn} \leq 1, \quad \text{Transition regime, slip flow;} \\ &\text{Kn} > 1, \quad \text{Free-molecule regime.} \end{aligned} \tag{1.7-4}$$

For example, if a piece of hypodermic tubing is used as a pressure probe, it might have an internal diameter as small as $L = 0.005$ in. $= 0.0127$ cm $\approx 1.3 \times 10^{-4}$ m. By using the data cited in Table 2.16-3, this would result in a Knudsen number $\text{Kn} = \lambda/L = 6.6 \times 10^{-8}$ at sea level and $\text{Kn} = 2 \times 10^{-3}/1.3 \times 10^{-4} \approx 15.0$ at an altitude of 75 km, which is generally taken to be the *reentry altitude* (i.e., the altitude above which the aerodynamic forces of the atmosphere on a vehicle are usually sufficiently small that they can be neglected). Thus, at sea level we expect continuum theory to be applicable for flow into the tube, whereas at 75 km free-molecule theory must be employed.

The vast literature on gas flow has emphasized the continuum regime. Patterson (1956; 1971) and Bird (1976) have applied nonequilibrium (non-Maxwellian) theory to a wide range of problems in the transition regime as well as in the free-molecule regime. In the transition regime the average tangential component of the molecular velocity is not zero, hence the name *slip flow*. Neither slip flow nor free-molecule theory is further considered in this work.

1.8 RELATION BETWEEN FLUID VISCOSITY AND MEAN-FREE-PATH LENGTH

In Book II, Section 9 of the *Principia*, Newton (1686) gave the first quantitative formulation of a law describing the action of fluid viscosity on a body. Newton considered a circular cylinder enclosed by a concentric cylindrical container with a liquid lying in the annular space between. He postulated that if the cylinder were rotated about its axis, it would generate a resistance (i.e., a torque) proportional to the difference in velocity between cylinder and wall. This idea has been extended to the idea of planar motion in a fluid of infinite lateral extent, resulting in what is called *Newton's law of resistance*.

The Macroscopic Law

Consider a layer of fluid resting on a horizontal surface; see Figure 1.8-1. On the top of the fluid is another, solid surface of which is shown only a portion, area A, which is dragged across the fluid* thereby exerting a horizontal (shear) force F on the fluid surface. The speed of the surface relative to the fixed bottom is Δu. The fluid in between has some velocity variation $u = u(y)$, which we initially treat as a linear profile.

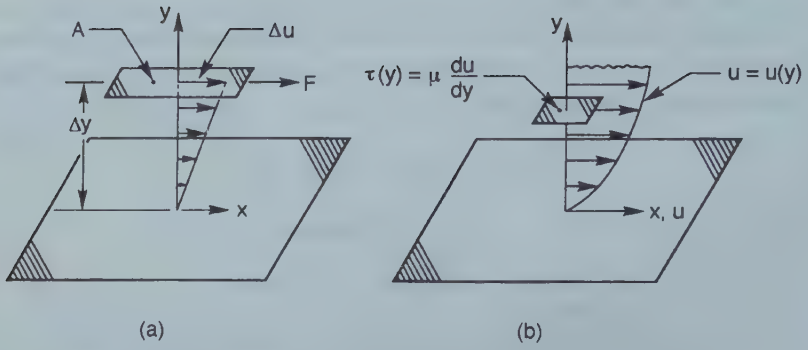

FIGURE 1.8-1 Relation of shear stress to velocity gradient in a flow: (a) linear velocity profile; (b) nonlinear velocity profile.

It is important to note that the flow velocity is a *directed velocity* on the macroscopic level, as distinguished from the microscopic, or molecular, random velocity of the molecules that is superimposed on the directed velocity. From the macroscopic view, the molecular velocity is not observable except in that the random motion produces certain macroscopic effects, such as pressure and, as it turns out, flow resistance.

Newton's law of resistance states that the resisting force is proportional to the area of the surface times the change in velocity divided by the thickness of the fluid layer, i.e.,

$$F \sim A \frac{\Delta u}{\Delta y}. \tag{1.8-1}$$

If we introduce the concept of *shear stress* (τ) and a factor of proportionality μ, called the *coefficient of dynamic viscosity*, then

$$\tau \equiv \frac{F}{A} = \mu \frac{\Delta u}{\Delta y}. \tag{1.8-2}$$

The units of μ are discussed in Section 1.10.

For the case in which the (directed) velocity profile is not linear, then the shear stress also becomes a function of the coordinate y. The local value can then be computed as a derivative,* i.e.,

$$\tau \equiv \lim_{\Delta y \to 0} \mu \frac{\Delta u}{\Delta y} = \mu \frac{du}{dy}. \tag{1.8-3}$$

* It has been demonstrated by countless experiments that on a material surface the relative, fluid mean velocity is zero except in the case of extremely rarified gases. This circumstance is referred to as the *no-slip condition.*

* This can be compared with the expression for shear stress for the more general case not restricted to motion parallel to the x-axis; then $\tau_{xy} = \mu(\partial v/\partial x + \partial u/\partial y)$.

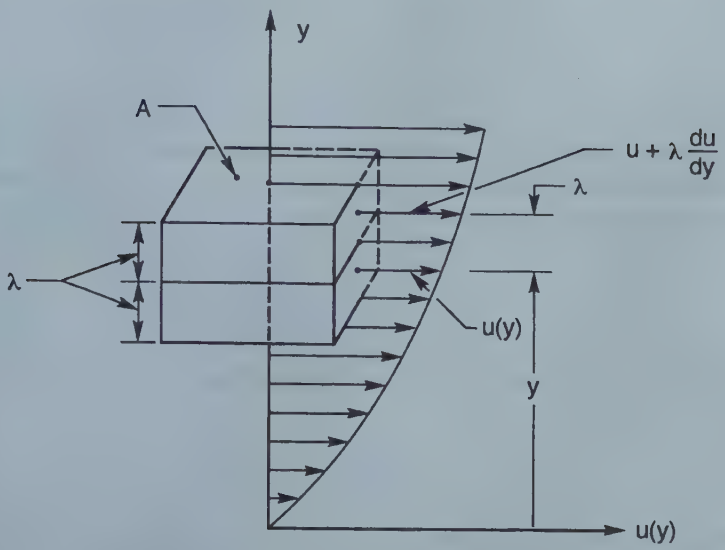

FIGURE 1.8-2 Schematic for calculation of viscosity coefficient.

A fluid that obeys Equation 1.8-3 is said to be a *newtonian* fluid. In a three-dimensional flow equation, Equation 1.8-3 must be replaced by a general relation between shear stress and the rate of strain in the medium (rate of fluid deformation).

Newton's law of friction is a *phenomenological** statement of the macroscopic behavior of the fluid. It is possible to formulate a comparable law from the molecular viewpoint. By comparing the two expressions, it is then possible to obtain an expression that relates the coefficient of dynamic viscosity to the molecular mean speed.

The Molecular Law

In a flow over a surface, let the directed velocity profile be given as $u = u(y)$. Now consider two adjacent layers of fluid each of thickness λ, the local mean-free-path length, see Figure 1.8-2. Then, the average directed velocity in each of the two layers is u, $u + (du/dy)\lambda$, respectively. Let the local number density be ν and the mean free speed be C. If we consider a pair of rectangular parallelepipeds of area A and height λ, interfacing at the common surface, then the time interval required for one molecule to traverse the height is $\Delta t = \lambda/C$. Since, on the average, one sixth of the molecules are moving toward any one face, the total number of molecules moving upward through A in interval Δt is

$$\tfrac{1}{6}\nu A\lambda = \tfrac{1}{6}\nu AC\Delta t.$$

After these molecules have interacted (collided) with the molecules in the upper layer, they must acquire the same average directed x-momentum as the upper layer. This change in momentum, which is equal to

* A relation depending only on quantities directly evident to the observer.

$$\tfrac{1}{6}\nu AC\Delta t\left(m\frac{du}{dy}\lambda\right)=\tfrac{1}{6}m\nu C\lambda\frac{du}{dy}A\Delta t,$$

exhibits itself macroscopically as a shear force exerted by the upper layer on the lower. Similarly, there is a transport of molecules from the upper to the lower layer resulting in a second, numerically identical, loss of momentum from the upper. Consequently, the shear stress exerted by the upper on the lower, corresponding to area A, is

$$\tau=\frac{F}{A}=\frac{\Delta(\text{Momentum})}{A\Delta t}=\frac{1}{3}m\nu C\lambda\frac{du}{dy}=\frac{1}{3}\rho C\lambda\frac{du}{dy}. \tag{1.8-4}$$

By comparison of Equations 1.8-3 and 1.8-4, we see that

$$\mu=\frac{1}{3}\rho C\lambda. \tag{1.8-5}$$

Although the preceding calculation is crude, it gives a result that is surprisingly good. The exact expression has been obtained by S. Chapman* as

$$\mu=0.499\rho C\lambda. \tag{1.8-6}$$

If we eliminate λ by means of Equation 1.7-2, then

$$\mu=\frac{0.499mC}{\pi\sqrt{2}\sigma^2}. \tag{1.8-7}$$

Since, according to Equation 1.6-8, $C \sim T^{1/2}$, then, for a perfect gas,

$$\mu \sim T^{1/2}. \tag{1.8-8}$$

The surprising fact that viscosity depends on temperature but not on pressure or density was first demonstrated by Robert Boyle in 1660** who found that the rate at which the oscillations of a pendulum suspended in a gas chamber died away remained the same even after the chamber was partially evacuated. The increase in viscosity of a gas with temperature is also confirmed by a vast array of experiments. In Section 1.11 a brief description is given of how Equation 1.8-8 is modified to account for the behavior of real gases.

1.9 ON HEAT CONDUCTION

In the preceding section we saw, from a molecular viewpoint, that the action of viscosity can be interpreted as the transport of momentum between layers of a fluid

* See Jeans (1952, p. 163).
** See Jeans (1952), p. 164, for the reference.

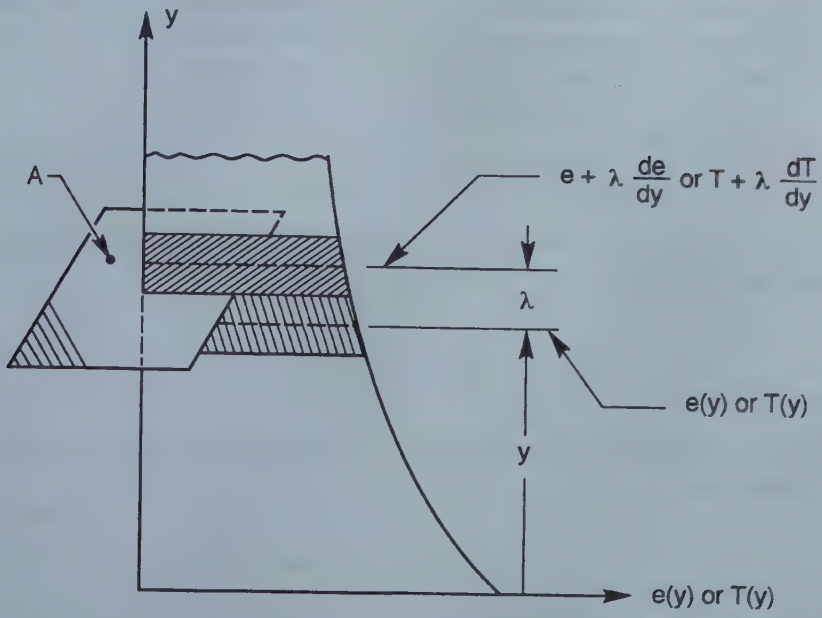

FIGURE 1.9-1 Schematic for relating heat flux term to temperature gradient.

in which gradients exist in the directed, or macroscopic, velocity. Similarly, we can show that if temperature gradients exist in a gas there is transport of thermal energy, or kinetic energy of random motion. The macroscopic term for this type of energy transport is *heat conduction*. Consider two adjacent layers of the gas, each of thickness λ, centered at y, $y + \lambda$, respectively, Figure 1.9-1.

From thermodynamic theory, the specific internal energy function for a thermally and calorically perfect gas (i.e., a gas which obeys Equation 1.6-7), and for which the coefficient of specific heat at constant volume c_v is constant) is given by $e = e(T) = c_v T$. Then, for a gas in which there is temperature gradient (or equivalently a gradient of the internal energy function) in the y-direction, the energy per molecule transported by molecular activity from a lower layer to an upper layer of mean separation λ is $-m(de/dy)\lambda$, m being the molecular mass. For all molecules transported across an area A, in both directions, over interval Δt, the energy gain of the upper layer is

$$Q = \frac{1}{3} \nu C(\Delta t) \left(-m \frac{de}{dy} \lambda \right) A.$$

The *energy flux* $\dot{q}$, equivalently the *heat flux*, is defined as the energy transported per unit area per unit time; i.e.,

$$\dot{q} \equiv \frac{Q}{A \Delta t} = -\frac{1}{3} m \nu C \lambda \frac{de}{dy} = -\frac{1}{3} \rho C \lambda \frac{de}{dy}.$$

Since our analysis is restricted to a perfect gas, we can eliminate the internal energy function in favor of the temperature. At the same time we can introduce Equation 1.8-5 to yield

$$\dot{q} = -\mu c_v \frac{dT}{dy}. \tag{1.9-1}$$

However, from *Fourier's law of heat conduction*, an empirical, macroscopic law, the heat flux is simply

$$\dot{q} = -k \frac{dT}{dy}, \tag{1.9-2}$$

where k is the *coefficient of heat conduction.** Comparison of Equations 1.9-1 and 1.9-2 shows that

$$k = \mu c_v. \tag{1.9-3}$$

Obviously, if Equation 1.9-3 holds, the ratio $\mu c_v/k$ is equal to unity, according to the simple model employed. It is precisely at this juncture that the elementary kinetic theory is least accurate. The accurate theory takes into account that a gas with a gradient in the directed velocity is not in equilibrium, and that is necessary to abandon the principle of equipartition of energy in favor of a more careful accounting of the distribution of energy between the translational and rotational degrees of freedom. This has resulted in *Eucken's formula*:**

$$k = \frac{9\gamma - 5}{4} \mu c_v, \tag{1.9-4}$$

where $\gamma \equiv c_p/c_v$ is the *ratio of specific heats* and is constant for a perfect gas.

It is next convenient to introduce a dimensionless parameter, the *Prandtl number* Pr, which appears in many problems in heat transfer. Its definition is

$$\mathrm{Pr} \equiv \mu c_p/k = \gamma \mu c_v/k, \tag{1.9-5}$$

which can be combined with Equation 1.9-4 to yield

$$\mathrm{Pr} = \frac{4\gamma}{9\gamma - 5}. \tag{1.9-6}$$

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* Some authors use the symbol k to denote the ratio of specific heats c_p/c_v.

** See Jeans (1952), p. 190, for the details.